The canonical formulation of $E_{6(6)}$ exceptional field theory

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Lars Thomas Kreutzer

Präsidentin der Humboldt-Universität zu Berlin
Prof. Dr.-Ing. Dr. Dr. h.c. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät
Prof. Dr. Elmar Kulke

Gutachter/innen:	1. Prof. Dr. Dr. h.c. Hermann Nicolai
	2. Prof. Dr. David Berman
	3. Dr. Olaf Hohm
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Declaration of independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

Signed on the 2nd of June, 2021, Berlin *Lars Thomas Kreutzer*

Abstract

In 1978 Eugène Cremmer and Bernard Julia discovered the existence of hidden non-compact global $E_{n(n)}(\mathbb{R})$ exceptional symmetries in the maximal supergravity theories that follow from the compactification of eleven-dimensional supergravity on an n-torus [1, 2]. The existence of these hidden exceptional symmetries in maximal supergravity theories is one of their most notable features, but the role of these symmetries is not yet fully understood at the quantum level. Moreover it has only been known since 2013 how a manifestly $E_{n(n)}$ covariant exceptional field theory (ExFT) can be constructed, which is based on an extended generalised exceptional geometry and in particular contains the eleven-dimensional supergravity [3].
In this thesis we construct and investigate the canonical formulation of the (bosonic) $E_{6(6)}$ ExFT, which can be seen as the starting point of the canonical quantisation procedure. We calculate the explicit non-integral form of the topological term of the $E_{6(6)}$ ExFT and explore a topological model theory based on the two-form kinetic term, where we identify problems regarding the construction of a Dirac bracket in an extended generalised geometry. To illustrate the construction of an extended generalised geometry we explicitly construct the Y-tensor for the symplectic group $\mathrm{Sp}(2n)$. Furthermore we establish a simplified canonical treatment of the scalar coset constraints, which we illustrate for the $\mathrm{SL}(n)/\mathrm{SO}(n)$ coset. As a preparation to the canonical analysis of the $E_{6(6)}$ ExFT we calculate the canonical formulation of the manifestly $E_{6(6)}$ invariant ungauged maximal five-dimensional supergravity theory and carry out a comprehensive canonical analysis including all gauge transformations and the full constraint algebra. We then proceed to work out the canonical formulation of the $E_{6(6)}$ ExFT. We calculate the full ExFT Hamiltonian, most of the canonical (gauge) transformations and parts of the constraint algebra. Moreover we examine how the canonical formulation can be expressed in the generalised vielbein form and we discuss the possible existence of generalised Ashtekar variables.

Keywords: supergravity, duality covariance, exceptional field theory, exceptional geometry, canonical formalism, Hamiltonian formalism, Hamiltonian

Zusammenfassung

Eugène Cremmer und Bernard Julia haben 1978 die Existenz von verborgenen nichtkompakten globalen $\mathrm{E}_{n(n)}(\mathbb{R})$-exzeptionellen Symmetrien in den maximalen Supergravitationstheorien, die aus der Kompaktifizierung der elfdimensionalen Supergravitation auf einem n-Torus folgen, entdeckt [1, 2]. Die Existenz dieser exzeptionellen Symmetrien in den maximalen Supergravitationstheorien ist eine ihrer bemerkenswertesten Eigenschaften, aber die Bedeutung dieser Symmetrien in der Quantentheorie ist noch nicht vollständig verstanden. Zudem ist erst seit 2013 bekannt, wie eine manifest $\mathrm{E}_{n(n)}$-kovariante exzeptionelle Feldtheorie (ExFT), die auf einer verallgemeinerten exzeptionellen Geometrie basiert und insbesondere die elfdimensionale Supergravitation beinhaltet, konstruiert werden kann [3].
In dieser Dissertation konstruieren und untersuchen wir die kanonische Formulierung der (bosonischen) $\mathrm{E}_{6(6)}$-ExFT, was als Ausgangspunkt der kanonischen Quantisierung angesehen werden kann. Wir ermitteln die explizite nicht-integrale Form des topologischen Terms der $\mathrm{E}_{6(6)}$-ExFT und untersuchen eine topologische Modelltheorie, die auf dem kinetischen Term der zwei-Form basiert, wobei wir Schwierigkeiten identifizieren, die die Konstruktion einer Dirac-Klammer auf einer verallgemeinerten exzeptionellen Geometrie betreffen. Um die Konstruktion einer verallgemeinerten Geometrie zu illustrieren konstruieren wir explizit den Y-Tensor für die symplektische Gruppe $\mathrm{Sp}(2n)$. Außerdem beschreiben wir eine vereinfachte kanonische Behandlung der Zwangsbedingungen des skalaren symmetrischen Raumes, welche wir für den symmetrischen Raum $\mathrm{SL}(n)/\mathrm{SO}(n)$ erläutern. Zur Vorbereitung der kanonischen Analyse der $\mathrm{E}_{6(6)}$-ExFT untersuchen wir die kanonische Formulierung der manifest $\mathrm{E}_{6(6)}$-invarianten ungeeichten maximalen fünfdimensionalen Supergravitationstheorie und führen eine umfassende kanonische Analyse, inklusive aller Eichtransformationen und der vollständigen Poisson-Algebra der Zwangsbedingungen, durch. Anschließend fahren wir mit der Ermittlung der kanonischen Formulierung der $\mathrm{E}_{6(6)}$-ExFT fort. Wir errechnen die vollständige Hamilton-Funktion der ExFT, sowie den Großteil der kanonischen (Eich-)Transformationen und Teile der Poisson-Algebra der Zwangsbedingungen. Zudem untersuchen wir, wie die kanonische Formulierung durch das verallgemeinerte Vielbein ausgedrückt werden kann und erörtern die mögliche Existenz von verallgemeinerten Ashtekar-Variablen.

To my parents.

"ὁ κόσμος, ἀλλοίωσις· ὁ βίος, ὑπόληψις."

"Mundus mūtātiō, vīta opīniō."

— Marcus Aurelius Antoninus Augustus,
τὰ εἰς ἑαυτὸν ἠθικά (Meditations), IV, 3

"But among the principles readiest to thine hand, upon which thou shalt pore, let there be these two. One, that objective things do not lay hold of the soul, but stand quiescent without; while disturbances are but the outcome of that opinion which is within us. A second, that all this visible world changes in a moment, and will be no more; and continually bethink thee to the changes of how many things thou hast already been a witness. 'The Universe—mutation: Life—opinion.'" [4]

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Contents

Chapter 1

Introduction

More than a century ago Albert Einstein developed the theory of general relativity, which is a classical (i.e. non-quantum) theory that describes gravity in an entirely geometric form [5]. The space-time geometry is described by the dynamical metric field $g_{\mu\nu}$. Moreover the theory is generally covariant, which means that the Einstein field equations (1.1), that govern the dynamics of the metric field, are the same in all coordinate systems.

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = \frac{8\,\pi\, G_N}{c^4}\, T_{\mu\nu} \tag{1.1}$$

The left hand side of the Einstein field equations is fully geometric and only depends on the metric field and its curvature. The right hand side however describes the energy or matter content of the universe. Albert Einstein himself already noted in 1916 that, because of the quantum properties of matter, this classical and deterministic theory of gravity would necessarily require a modification [6]. This leads to the question of what such a theory of quantum gravity should look like.[1]

In the Einstein field equations G_N is Newton's constant and c is the speed of light in vacuum. Combined with the reduced Planck's constant $\hbar$ we can use these quantities to define a mass or energy scale by $m_{\mathrm{P}} := \sqrt{\hbar\, c/G_N} \approx 2.18 \cdot 10^{-8}$ kg, which is called the Planck mass.[2] Similarly we can define the Planck length $l_{\mathrm{P}} := \sqrt{\hbar\, G_N/c^3} \approx 1.62 \cdot 10^{-35}$ m and the Planck time $t_{\mathrm{P}} := \sqrt{\hbar\, G_N/c^5} \approx 5.39 \cdot 10^{-44}$ s. Naively one would expect to see the effects of quantum gravity only at energies around or above the Planck mass, but even the most advanced particle physics experiments at the Large Hadron Collider (LHC) only reach a total kinetic energy that is equivalent to a mass of ca. 10^{-23} kg and even some of the most energetic cosmic radiation that has been measured on earth only reaches an energy scale equivalent to ca. 10^{-16} kg [9], which is far less than the Planck mass. Equivalently one should expect to have to observe processes that take place at scales below the Planck length or Planck time scales in order to see any measurable effects of quantum gravity.

Due to this immense difference between the Planck scale and the scales that are accessible to current experiments and observations there is no real phenomenological or experimental guidance as to what a theory of quantum gravity should look like. The only physical phenomena for which the quantum effects of gravity are currently (generally) expected to be relevant are the space-time singularities found at the centre of black holes and at the Big Bang, where general relativity is ill-defined. Unfortunately such singularities are also conjectured to be shielded from any direct observation (this is known as the cosmic censorship hypotheses) [10].

In absence of any direct measurements many different approaches to quantum gravity have been developed, some of which take a minimalistic approach to extending

[1]Reference [7] is a good general introduction to quantum gravity and reference [8] contains an interesting description of some of the historical developments concerning quantum gravity.

[2]In the remainder of this thesis we will work in Planck units with $\hbar = c = G_N = 1$.

general relativity, while others introduce radically new concepts, such as the idea of replacing point-like fundamental particles with one-dimensional fundamental relativistic strings, which is known as string theory.[3] In order to be consistent with the known physics, any theory of quantum gravity needs to reduce to (or at least contain) general relativity at energy scales below the Planck scale and be compatible with all other experimentally verified results. One can try to quantise general relativity perturbatively around a flat space-time, but one finds that the theory is non-renormalisable, which means that one cannot remove the divergencies (or infinities), that appear in this case at the second order in the perturbation theory, without introducing an infinite number of counter terms, which in turn lead to an infinite number of constants that would need to be determined experimentally and this would essentially make the theory lose all its predictive power [12]. There are numerous other computational and conceptual problems that arise when considering theories of quantum gravity, such as the meaning of quantum fluctuations of the causal structure of space-time and the role of time itself (this is known as the problem of time) [13, 14].

One possible approach to quantum gravity, which we describe in more detail in chapter 2, is called supergravity and it postulates an enlarged local space-time symmetry of general relativity that is called supersymmetry [15]. One particular supergravity theory (the four-dimensional maximal $\mathcal{N} = 8$ theory [16]) could moreover be a perturbatively finite theory of quantum gravity [17, 18]. The divergencies that arise in the perturbative quantisation of general relativity are (at least in part) cancelled in supergravity due to the additional supersymmetry between the bosonic ("forces") and fermionic ("matter") parts of the theory.

One can also consider supergravity theories in more than four dimensions and although there is currently no experimental evidence for the existence of extra dimensions, it can be mathematically useful to consider higher dimensional space-time geometries and to relate them back to lower dimensional physics. For example one finds that for an eleven-dimensional space-time geometry there exists only one unique supergravity theory [19]. If one assumes that seven of the dimensions of the eleven-dimensional space-time are given by a compact 7-torus and furthermore that the size of this 7-torus is very small, e.g. comparable to the Planck length l_{P}, then the eleven-dimensional theory can be equivalently described in terms of the above-mentioned four-dimensional supergravity theory for energy scales that are well-below the Planck scale. This procedure is called (toroidal) compactification and in this manner one can recover a four-dimensional theory from the eleven-dimensional supergravity theory.

In 1978 it was discovered by Eugène Cremmer and Bernard Julia that an additional hidden (global) symmetry emerges in the four-dimensional (maximal) supergravity theory that is obtained from the eleven-dimensional supergravity by toroidal compactification and this symmetry is described by the (non-compact real) exceptional Lie group $\mathrm{E}_{7(7)}(\mathbb{R})$ [1, 2]. More generally one can find an $\mathrm{E}_{n(n)}(\mathbb{R})$ symmetry in the n-torus compactifications of eleven-dimensional supergravity. The unexpected existence of these hidden exceptional symmetries lead to the question of whether this symmetry already exists in the eleven-dimensional supergravity. In the following decade reformulations of the eleven-dimensional supergravity were found that manifestly exhibit symmetries that are subgroups of the exceptional groups [20–22].

[3] Reference [11] tries to visually represent some of the conceptual relations between various approaches to quantum gravity and reference [8] contains a visual representation of the historical development of some concepts.

It was however only in 2013 that the first fully and manifestly $\mathrm{E}_{n(n)}(\mathbb{R})$ covariant field theory, called exceptional field theory (ExFT), was constructed, which in particular contains the dynamics of eleven-dimensional supergravity [3]. ExFT achieves manifest (local) $\mathrm{E}_{n(n)}(\mathbb{R})$ covariance with the help of a generalised and extended notion of space-time geometry, which was developed in the previous decade and which we describe in more detail in chapter 3. This generalised exceptional geometry is moreover related to the concept of geometrisation, which is the idea that one can find a common higher dimensional geometric origin for originally non-geometric degrees of freedom. This idea was historically pioneered by Theodor Kaluza and Oskar Klein in the 1920s, when it was discovered that four-dimensional general relativity coupled to Maxwell theory and a massless scalar field can be equivalently described by a purely gravitational theory in five dimensions when the five-dimensional theory is compactified on a circle [23, 24]. Similarly ExFT combines geometric and non-geometric degrees of freedom of eleven-dimensional supergravity. What is moreover remarkable about ExFT is that the exceptional symmetry seems to be intricately connected to supersymmetry, because the bosonic ExFT, without the requirement of supersymmetry, is already uniquely determined by the requirement of (generalised) diffeomorphism invariance [3, 25]. Nonetheless one can extend the ExFT to be supersymmetric [26]. This may hint at a more fundamental interpretation of the generalised exceptional geometry. A brief overview of some of the application of ExFT can be found in chapter 3.

The quantisation of ExFT poses several challenges due to the generalised exceptional geometry. The path integral formalism cannot be used to quantise the theory because any integral over the generalised exceptional geometry has to obey, what is called, the section condition (1.2).

$$Y^{KL}{}_{MN}\ \partial_K \otimes \partial_L = 0 \tag{1.2}$$

The section condition (1.2) restricts the coordinate dependence of any function on the generalised exceptional geometry in a way that is governed by the Y-tensor $Y^{KL}{}_{MN}$, which is determined by the structure of the exceptional group $\mathrm{E}_{n(n)}(\mathbb{R})$. The coordinate derivatives ∂_K sit in a representation of $\mathrm{E}_{n(n)}(\mathbb{R})$. In ordinary differential geometry there is no analogue to the section condition of extended generalised geometry. In order to quantise ExFT one therefore has to make use of other kinds of quantisation procedures. One possibility is to consider the canonical quantisation procedure, which proceeds via the canonical (or Hamiltonian) formulation of the theory [27, 28]. The aim of this thesis is the construction and analysis of the canonical formulation of ExFT, which can be seen as the first step towards the canonical quantisation of ExFT.

The canonical formulation of double field theory [29, 30], which is based on an extended generalised $\mathrm{O}(n, n)$ geometry (doubled geometry) similar to the geometry of ExFT, has been examined in [31]. Very recently the geometric quantisation procedure has been discussed in the context of double field theory [32]. However the geometric quantisation approach may be less suitable in the context of ExFT as there is no clear analogy between the extended generalised exceptional geometry and the geometry of phase space — the quantisation of ExFT is briefly commented on in [32]. For special geometries, which are of the form of Minkowski space times a compact torus, some $\mathrm{E}_{n(n)}$ invariant amplitudes of ExFT have been computed (up to three loops) in [33–35].

To simplify the canonical analysis we focus on the bosonic sector of the $\mathrm{E}_{6(6)}$ ExFT [3, 25, 36], which in contrast to some of the other ExFTs does not involve any self-dual differential forms or constrained compensator fields, that would otherwise complicate the canonical analysis. Some of the main challenges in the canonical analysis of the $E_{6(6)}$ ExFT are the treatment of the generalised exceptional geometry and the complicated topological term.

Apart from the aim of the canonical quantisation of ExFT, one may furthermore hope to learn about some other aspects of the generalised exceptional geometry from the canonical formulation of ExFT — such as the role and physical meaning of the section condition in the canonical formulation of ExFT or how the generalised diffeomorphism transformations are generated canonically. Moreover it may be interesting to investigate the local initial value problem for the extended generalised exceptional geometry. Furthermore one could try to identify a generalised definition of asymptotic flatness and ADM charges for non-compact extended generalised exceptional geometries, in analogy to what has been done in the canonical formulation of double field theory in [31]. The Ashtekar connection is an alternative phase space variable in general relativity that transforms the canonical constraints into polynomial expressions [37, 38], due to the results of [39] one might hope to find the definition of a generalised Ashtekar connection for the generalised exceptional geometry from the canonical formulation of ExFT — this is discussed in more detail in chapter 7.

The three central concepts used in this thesis are supergravity, generalised (exceptional) geometry and the canonical formalism. Each of these topics is discussed respectively in the chapters 2, 3 and 4. In chapter 5 the canonical analysis of the five-dimensional $E_{6(6)}$ invariant (maximal) supergravity theory is carried out as an intermediate step — this theory is related to the $E_{6(6)}$ ExFT by the trivial $\partial_K \equiv 0 \; \forall K$ solution of the section condition (1.2). In chapter 6 we construct and analyse the canonical formulation of the $E_{6(6)}$ ExFT itself. A more detailed outline of this thesis can be found in section 1.1.

1.1 Outline

The outline of this thesis is as follows:

In chapter 2 the concept of supersymmetry is introduced and the basic ideas of supergravity are discussed. We explain the concept of toroidal compactifications and describe the global exceptional symmetries that arise in the compactifications of eleven-dimensional supergravity on an n-torus. Furthermore we examine how group invariant supergravity Lagrangians can be constructed and how subgroups of global symmetries can be gauged. The relationship between hidden exceptional symmetries of maximal supergravity and string theory dualities is briefly discussed.

In chapter 3 the concept of geometrisation is first illustrated by looking at Kaluza-Klein theory. It is then explained how generalised notions of geometry can be used to achieve the geometrisation of degrees of freedom in more general settings. We briefly discuss generalised (complex) geometry, which is based on the idea of an enlarged tangent bundle and then look at doubled geometry, which moreover extends the dimension of the base geometry at the cost of introducing a consistency condition called the section condition. Next we describe how the string theory low energy effective action (of the NSNS sector) can be rewritten in a manifestly duality covariant and geometrised formulation called double field theory. Moreover we examine the construction of a generalised symplectic geometry as an example and explicitly construct the Y-tensor for the symplectic group $\mathrm{Sp}(2n)$. We then review how an extended generalised exceptional geometry can be used to reformulate the dynamics of eleven-dimensional supergravity in a manifestly $\mathrm{E}_{6(6)}$ covariant form called exceptional field theory and we construct the explicit non-integral (not manifestly gauge invariant) form of the topological term of this theory.

In chapter 4 we first review some of the basics of the canonical (or Hamiltonian) formalism of constrained Hamiltonian systems and then briefly discuss some aspects of the Arnowitt-Deser-Misner formulation of general relativity. In order to examine and illustrate an explicit and implicit canonical treatment of coset constraints we look at the canonical formulation of the $\mathrm{SL}(n)/\mathrm{SO}(n)$ scalar coset sigma model. Next we investigate the canonical formulation of a topological two-form model theory on an extended generalised geometry, which is based on the topological term of the $\mathrm{E}_{6(6)}$ exceptional field theory. We construct the canonical constraint algebra and discuss some problems regarding the construction of a Dirac bracket in this model.

In chapter 5 we investigate the canonical formulation of the bosonic sector of the unique manifestly $\mathrm{E}_{6(6)}$ invariant ungauged maximal five-dimensional supergravity theory, which is closely related to the $\mathrm{E}_{6(6)}$ exceptional field theory by the trivial solution of the section condition. We calculate the canonical Hamiltonian, the complete set of canonical constraints, all gauge transformations and construct the complete Poisson bracket algebra of the canonical constraints. In this analysis we identify a crucial redefinition of the canonical variables and examine the canonical treatment of the topological term.

In chapter 6 we investigate the canonical formulation of the bosonic sector of the $\mathrm{E}_{6(6)}$ exceptional field theory. We carry out the Legendre transformation and calculate the canonical Hamiltonian of the theory. The canonical constraints are identified and we calculate most of the canonical (gauge) transformations and part of the canonical constraint algebra. Furthermore we discuss the treatment of the intricate topological term and examine the role of the section condition in the canonical formalism. Finally we introduce the internal generalised vielbein and explain how the canonical formulation of exceptional field theory can be rewritten in terms of these variables.

In chapter 7 we review and discuss the main findings of this thesis. Furthermore we give an outlook on possible applications of the results of this thesis, including on the possible existence of generalised Ashtekar variables and the quantisation of exceptional field theory.

In appendix A we list some useful mathematical identities concerning the Levi-Civita symbol, the Dirac delta distribution and derivatives of the vielbein determinant.

In appendix B additional Poisson bracket relations and intermediate results concerning the content of chapter 5 are listed.

1.2 Publications

The results presented in this thesis are mainly based on the following publications:

I Lars T. Kreutzer, “Canonical analysis of $E_{6(6)}(\mathbb{R})$ invariant five dimensional (super-)gravity”, *Journal of Mathematical Physics* 62, p. 032302 (2021), DOI: 10.1063/5.0037092, arXiv: 2005.13553, cited in the following as [40]

II Lars T. Kreutzer, “The canonical formulation of $E_{6(6)}$ exceptional field theory”, arXiv: 2105.02238 (submitted), cited in the following as [41]

Section 4.1, section 4.2, section 4.3, chapter 5, appendix A and appendix B are based on and closely follow the structure of the publication [40]. Section 3.5, section 4.4 and chapter 6 are based on and closely follow the structure of the publication [41]. Chapter 7 is based on parts of both publications [40, 41], but more closely follows the structure of parts of [41]. Many of the sections listed above contain additional (unpublished) explanations and information that are not contained in the publications [40, 41]. The results of section 3.4, which are derived from the procedure described in [42, 43], have not been published elsewhere.

Chapter 2

Supergravity

In this chapter we briefly introduce supersymmetry and supergravity, we explain the relation of supergravity to superstring theory and the emergence of exceptional symmetries in supergravity.

2.1 What is supersymmetry?

In order to understand what supergravity is we first need to define supersymmetry. Supersymmetry (SUSY) is an extension of the Poincaré space-time symmetry.[1] The Poincaré Lie algebra is generated by the generators of d-dimensional translations P_μ, the generators of the $\mathrm{SO}(1, d-1)$ Lorentz transformations $L_{\mu\nu}$ and the generators X_M of some internal symmetry. The Coleman-Mandula no-go theorem, published in 1967 [45], states that the most general symmetry Lie algebra of the scattering matrix of a physical theory can only combine the space-time symmetry with the internal symmetry in a trivial way $[X_M, P_\mu] = 0$, $[X_M, L_{\mu\nu}] = 0$ and that there is no mixing of particles of different spins.

One possible workaround to the Coleman-Mandula theorem is to relax the condition for the resulting algebra to be a Lie algebra. Instead one can consider a superalgebra with a $\mathbb{Z}_2$ graded product, which allows for so called fermionic generators Q^{A}_α (grading $n_a = 1$) in a spinor representation of the Lorentz group, in contrast to the bosonic generators P_μ, $L_{\mu\nu}$ and X_M (grading $n_a = 0$). The graded product of this super-Poincaré algebra is given by (2.1) where O_a is a generator of the superalgebra and f^c_{ab} are the structure constants.

$$\{O_a, O_b\} := O_a \cdot O_b - (-1)^{n_a n_b}\, O_b \cdot O_a = f^c_{ab}\, O_c \tag{2.1}$$

The super-Poncaré algebra is the supersymmetric extension of the Poincaré algebra. The Haag-Łopuszański-Sohnius no-go theorem, published in 1975 [46], extends the Coleman-Mandula theorem and also considers superalgebras. When acting on the fields of a theory the generators Q^{A}_α exchange fermionic and bosonic fields and therefore this transformation mixes different spins. In particular one finds that the product of two fermionic generators has to result in a translation (2.2) [46].[2]

$$\{Q^{\mathrm{A}}_\alpha, \bar{Q}^{\mathrm{B}}_{\dot\beta}\} = \delta^{AB}\, {\sigma^\mu}_{\alpha\dot\beta}\, P_\mu \tag{2.2}$$

Therefore a consistent theory that is invariant under local supersymmetry is necessarily invariant under diffeomorphisms and thus it is a theory of gravity. Likewise it

[1] See reference [44] for lecture notes on supersymmetry.

[2] The relation (2.2) is written here for the four-dimensional case, but a similar relation holds in any dimension. The overline indicates the Dirac adjoint, the dots indicate the chirality (van der Vaerden notation) and the σ^μ are the Pauli matrices.

is true that a supersymmetric theory of gravity is necessarily invariant under local supersymmetry. The fermionic generators Q_α^{A} carry one spinor index α and the index A labels the different generators $\mathrm{A} = 1, \ldots, \mathcal{N}$. The number $\mathcal{N}$ is sometimes referred to as the number of supersymmetries. The components of the fermionic generators are also called supercharges. The case $\mathcal{N} = 1$ is referred to as basic supersymmetry, whereas $\mathcal{N} > 1$ is called extended supersymmetry. The maximal (real) number of supercharges that a theory can have without necessarily leading to fields of spin greater than two and without having several fields of spin two (gravitons) is 32 and this case is referred to as maximal supersymmetry (e.g. $\mathcal{N} = 8$ in $d = 4$ or $\mathcal{N} = 1$ in $d = 11$) [47]. The representation spaces of these groups are called supermultiplets.

2.2 What is supergravity?

Supergravity (SUGRA) theories are theories with *local* supersymmetry.[3] Therefore supergravity theories are necessarily also theories of supersymmetric gravity (cf. equation (2.2)). One possible motivation for considering theories of supersymmetric gravity is the hope that supersymmetry might cure the inherent problems (ultraviolet divergencies) that arise when quantising gravity, because supersymmetry seems to improve the quantum behaviour of theories with global supersymmetry, such as Super-Yang-Mills theory, Super-Maxwell theory or Wess-Zumino theory [51, 52]. Furthermore maximal ($\mathcal{N} = 8$) supergravity could be a finite quantum field theory that includes gravity [16–18]. Moreover it is in itself a remarkable fact that it is at all possible to write down a consistent theory that extends general relativity in a supersymmetric way — therefore implying a non-trivial spinorial structure that is hidden in general relativity.

The simplest supergravity theory has basic supersymmetry $\mathcal{N} = 1$ and the corresponding supermultiplet contains only one spin two field, the graviton (or metric) $G_{\mu\nu}$ and one spin 3/2 field, the gravitino ${\Psi_\mu}^\alpha$ [15]. The metric $G_{\mu\nu}$ can be used to define the (equivalent) vielbein ${E_\mu}^a$ by $G_{\mu\nu} = {E_\mu}^a\, {E_\nu}^b\, \eta_{ab}$, where η_{ab} is the Minkowski metric.[4] The gravitino is the supersymmetry partner of the graviton and the supersymmetry transformations (see equations (2.6) and (2.7)) exchange these fields [15]. We can write an action for this theory as in equation (2.3).

$$S_{\mathrm{SUGRA}} = \int d^d x \, (\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{RS}} + \mathcal{L}_{\mathrm{Int.}}) \tag{2.3}$$

The first part of the action is simply the Einstein-Hilbert term of general relativity (2.4), which is the kinetic term of the graviton [53–55]. E is the determinant of the vielbein ${E_\mu}^a$ and ${R_{\mu\nu}}^{ab}$ is the Riemann tensor.

$$\mathcal{L}_{\mathrm{EH}} = -\frac{E}{4}\, {E_a}^\mu\, {E_b}^\nu\, {R_{\mu\nu}}^{ab} \tag{2.4}$$

The second term in the action is the Rarita-Schwinger term (2.5), which was first published in 1941 [56].

$$\mathcal{L}_{\mathrm{RS}} = -\frac{i}{2}\, E\, \overline{\Psi}_\mu\, \gamma^{\mu\nu\rho}\, D_\nu \Psi_\rho \tag{2.5}$$

[3]See reference [15] for a comprehensive textbook introduction to supergravity, which contains most topics covered in this chapter. For lecture notes on supergravity see [48–50].

[4]The vielbein ${E_\mu}^a$ is introduced in detail in section 6.2. Note that the index a denotes a d-dimensional Lorentz index in this section, whereas a different convention for the indices is used in section 6.2.

It is the relativistic kinetic term for a spin 3/2 field in a curved background that is minimally coupled to gravity via the covariant derivative D_μ, which contains the spin connection. Because we are only interested in a qualitative discussion of the fermionic degrees of freedom, we will refrain from defining all the fermionic conventions and objects, instead we refer the reader to reference [15] for the details about the fermionic fields. The Lagrangian $\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{RS}}$ is called the universal part of the supergravity action because it is present in all supergravity actions [15]. The Einstein-Hilbert term effectively contains an infinite number of interaction terms for the graviton. The Rarita-Schwinger term is the kinetic term of a spinor coupled to gravity, but in general further higher order fermionic interaction terms $\mathcal{L}_{\text{Int.}} \sim \Psi^4$ are needed for the action to be supersymmetric. Using the supersymmetry transformations (2.6) and (2.7) with parameter ε, we can nonetheless verify that, at least at the lowest order in fermions, the universal action of supergravity is indeed supersymmetric (2.8) [15, 57].

$$\delta_\varepsilon E_\mu{}^a = -i\,\bar{\varepsilon}\,\gamma^a\,\Psi_\mu \tag{2.6}$$

$$\delta_\varepsilon \Psi_\mu = D_\mu \varepsilon \tag{2.7}$$

The result (2.8) holds in all dimensions to lowest order in the fermionic fields — in general however one needs to add very specific interaction terms and a very specific matter content to make the theory supersymmetric to all orders.

$$\delta_\varepsilon(\mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{RS}}) = 0 \tag{2.8}$$

The universal supergravity action for the case of four-dimensional basic supergravity (i.e. $d = 4$, $\mathcal{N} = 1$) is supersymmetric to all orders if the spin connection $\omega_{\mu ab}$ is modified to include an additional gravitino contorsion term $\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}$, with the contorsion tensor defined by $K_{\mu\nu\rho} := -\frac{1}{4}(\overline{\Psi}_\mu\gamma_\rho\Psi_\nu - \overline{\Psi}_\nu\gamma_\mu\Psi_\rho + \overline{\Psi}_\rho\gamma_\nu\Psi_\mu)$, which effectively acts as a spinorial interaction term [15, 57].

2.2.1 M-theory: eleven-dimensional supergravity

In the case of maximal supergravity, with 32 supercharges, there is only one unique supermultiplet and one unique associated supergravity theory. In eleven dimensions there are already 32 supercharges for a single supersymmetry generator Q_μ and therefore there is a unique eleven-dimensional theory of supergravity which is sometimes referred to as M-theory, it was first published in 1978 [19]. Its canonical formulation was first published in 1986 [58, 59].

The bosonic field content of M-theory is just the graviton and a three-form $C_{\mu\nu\rho}$ with field strength $F_{\mu\nu\rho\sigma} := 4\,\partial_{[\mu}C_{\nu\rho\sigma]}$. The bosonic action is given by (2.9), it consists of the Einstein-Hilbert term, a three-form kinetic term F^2 and a topological term of the form $F \wedge F \wedge C$.

$$S = \int d^{11}x \left(E\,R_{11} - \frac{E}{48}\,F_{\mu\nu\rho\sigma}\,F^{\mu\nu\rho\sigma} + \frac{1}{12^4}\,\epsilon^{\mu_1\ldots\mu_{11}}\,F_{\mu_1\ldots\mu_4}\,F_{\mu_5\ldots\mu_8}\,C_{\mu_9\mu_{10}\mu_{11}} \right) \tag{2.9}$$

The relative coefficients in (2.9) are fixed by requiring the action to have a supersymmetric completion [15, 19]. The only fermionic field is the Rarita-Schwinger spinor (gravitino) $\Psi_\mu{}^\alpha$ and the fermionic action is given by the universal Rarita-Schwinger term plus $\mathcal{O}(\Psi^2\,F)$ interaction terms [2]. To simplify the discussion we will, in the following, only consider the bosonic sector of supergravity. The bosonic action (2.9) is invariant under diffeomorphisms (2.10) and (2.11), tensor gauge transformations

(2.12) and local SO(1,10) Lorentz transformations (2.13).

$$\delta_\xi E_\mu{}^a = \xi^\nu\,\partial_\nu E_\mu{}^a + \partial_\mu\xi^\nu\,E_\nu{}^a \tag{2.10}$$

$$\delta_\xi C_{\mu\nu\rho} = \xi^\sigma\,F_{\sigma\mu\nu\rho} \tag{2.11}$$

$$\delta_\zeta C_{\mu\nu\rho} = 3\,\partial_{[\mu}\zeta_{\nu\rho]} \tag{2.12}$$

$$\delta_\lambda E_\mu{}^a = \lambda^a{}_b\,E_\mu{}^b \tag{2.13}$$

2.3 Toroidal compactifications and emerging symmetries

The eleven-dimensional supergravity theory described in section 2.2.1 is furthermore notable because it is possible to derive from it most ungauged maximally supersymmetric supergravity theories in dimensions lower than eleven by the process of toroidal compactification — the notable exception being the type IIB supergravity [60, 61].[5] Toroidal compactification means that we take the eleven-dimensional space-time $\mathcal{M}_{11}$ to be split as $\mathcal{M}_{11} = \mathcal{M}_{11-n} \times \mathrm{T}^n$ into a lower dimensional Lorentzian manifold and a compact n-torus whose radii we assume to be small enough for us to be able to assume a low energy limit which effectively leads to a lower dimensional theory in $11 - n$ dimensions.

2.3.1 S^1 compactification and cylinder condition

To illustrate the process of toroidal compactification of supergravity we briefly look at the simpler case of a free massless scalar $\hat{\phi}(\hat{x}^{\hat{\rho}})$ in $d+1$ dimensions that we compactify on a 1-torus, i.e. a circle S^1, as $\mathcal{M}_{d+1} = \mathcal{M}_d \times \mathrm{S}^1$ [2, 62].[6] This example is instructive and we will come back to a similar situation in section 3.1 when talking about Kaluza-Klein theory.

We split the coordinates of the $(d+1)$-dimensional geometry as $\hat{x}^{\hat{\mu}} = (x^\mu, y)$ where y is the coordinate on the circle. Because the circle is compact we can express the scalar field $\hat{\phi}(\hat{x}^{\hat{\rho}})$ in terms of its Fourier series as in equation (2.14), where $n \in \mathbb{Z}$ labels the Fourier modes $\phi^{(n)}(x^\rho)$ and $R > 0$ is proportional to the radius of the circle.

$$\hat{\phi}(\hat{x}^{\hat{\rho}}) = \sum_{n\in\mathbb{Z}} \phi^{(n)}(x^\rho)\, e^{\frac{iny}{R}} \tag{2.14}$$

The equation of motion for the free massless scalar field $\hat{\phi}(\hat{x}^{\hat{\rho}})$ is the free wave equation given by the d'Alembert operator as in (2.15).

$$\partial_{\hat{\mu}}\partial^{\hat{\mu}}\hat{\phi}(\hat{x}^{\hat{\rho}}) = 0 \tag{2.15}$$

Inserting the Fourier decomposition (2.14) of the scalar field into the equation of motion (2.15) we find the set of equations (2.16).

$$\partial_\mu\partial^\mu\phi^{(n)} - \frac{n^2}{R^2}\,\phi^{(n)} = 0 \quad \forall n \in \mathbb{Z} \tag{2.16}$$

In the equations (2.16) we see the d-dimensional d'Alembert operator term, however because of the exponential factor in the Fourier series the $\partial_y\partial_y$ derivative in (2.15)

[5] The "ungauged" refers here to the global R-symmetry that relates different supercharges [15, 52]. This in contrast to the gauged supergravity that we discuss in section 2.3.4.

[6] The argument presented here mainly follows [62].

generates an additional mass term with masses $m^2 = n^2/R^2$. We can therefore interpret (2.16) as the equations of motion for an infinite number of massive scalar fields in d dimensions. The only exception to this is the zero mode $\phi^{(0)}$, which is massless and obeys the equation of motion of a free massless scalar (2.17), just like equation (2.15) but in one dimension lower.

$$\partial_\mu \partial^\mu \phi^{(0)}(x^\rho) = 0 \tag{2.17}$$

Because the masses are inversely proportional to the radius of the circle $m \sim 1/R$ we can make the masses arbitrarily large by shrinking (i.e. compactifying) the circle. Physically speaking this means that we can consider the low energy limit of this theory where the masses are much larger than the energy scale that we are interested in $m(R) \gg E$. In this limit the massive modes decouple from the theory and we can consistently truncate to the theory that only has the massless mode [62]. Equivalently we can impose the "cylinder condition" (or Kaluza-Klein condition) (2.18) which reduces the equation (2.15) directly to (2.17).

$$\partial_y \equiv 0 \tag{2.18}$$

The set of equations (2.16) in d-dimensions are equivalent to the original equation of motion (2.15) in $(d+1)$-dimensions. The truncated theory, described by just the equation (2.17) is the theory of a single free massless scalar field in d-dimensions and is related to the higher dimensional theory by circular compactification.

2.3.2 Emerging exceptional symmetries $\mathbf{E}_{n(n)}$

Toroidal compactification proceeds in analogy to the circular reduction of section 2.3.1 and for theories other than that of a free scalar the resulting lower dimensional theory can be very interesting.[7] In general the lower dimensional theory will inherit a subgroup of the symmetries of the higher dimensional theory. Which subgroup this is depends on the compact geometry (which can even be more general than a torus, see e.g. [63, 64]) and the low energy limit that is chosen [2, 65].

For the $\mathcal{M}_{11} = \mathcal{M}_d \times \mathrm{T}^n$ (with $d := 11 - n$) torus compactification of eleven-dimensional supergravity the eleven-dimensional fields $E_\mu{}^a$ and $C_{\mu\nu\rho}$ split into d-dimensional tensors with additional internal indices and the finite number of massless modes that survive become the field content of the d-dimensional theory. In this section we will not look at the details of this reduction but instead focus on the results. In chapter 3 we will come back to this point and describe how the fields of eleven-dimensional supergravity decompose in this reduction and how one can arrive at the compactified theory on an alternative path.

In general the compactification of eleven-dimensional supergravity on an n-torus T^n is equivalent to the ungauged maximal supergravity theory in d-dimensions (cf. [1, 2, 66, 67]). In 1978 it was discovered by Eugène Cremmer and Bernard Julia that there are additional hidden non-compact global exceptional symmetries $\mathrm{E}_{n(n)}$ present in the lower dimensional theories that are realised non-linearly [1, 2, 66, 67]. Furthermore there are associated local symmetries given by the maximal compact subgroup of $\mathrm{E}_{n(n)}$.
The notion of exceptional Lie group (in the split real form) $\mathrm{E}_{n(n)}$ that is used here is

[7] In this thesis the terms compactification and reduction are used synonymously.

n	$\mathrm{E}_{n(n)}$	$\dim(\mathrm{E}_{n(n)})$	$\mathrm{K}(\mathrm{E}_{n(n)})$	$\dim(\mathrm{K}(\mathrm{E}_{n(n)}))$
1	GL(1)	1	SO(1)	0
2	$\mathrm{SL}(2)\times\mathbb{R}$	4	SO(2)	1
3	$\mathrm{SL}(3)\times\mathrm{SL}(2)$	11	$\mathrm{SO}(3)\times\mathrm{SO}(2)$	4
4	SL(5)	24	SO(5)	10
5	SO(5,5)	45	$(\mathrm{SO}(5)\times\mathrm{SO}(5))/\mathbb{Z}_2$	20
6	$\mathrm{E}_{6(6)}$	78	$\mathrm{USp}(8)/\mathbb{Z}_2$	36
7	$\mathrm{E}_{7(7)}$	133	$\mathrm{SU}(8)/\mathbb{Z}_2$	63
8	$\mathrm{E}_{8(8)}$	248	$\mathrm{SO}(16)/\mathbb{Z}_2$	120

Table 2.1: The split real forms of the exceptional groups $\mathrm{E}_{n(n)}$, their maximal compact subgroups $\mathrm{K}(\mathrm{E}_{n(n)})$ and their dimensions. This table is based on [70], an early version of this list can be found in [66].

a generalisation of the exceptional groups E_6, E_7, E_8 from the Cartan classification, by choosing an analogous form for the Dynkin diagrams of the algebras [15, 68, 69]. The groups $\mathrm{E}_{n(n)}$ and their maximal compact subgroups $\mathrm{K}(\mathrm{E}_{n(n)})$ are listed in table 2.1.

The discovery of the hidden exceptional symmetries in lower dimensions lead to the question of what the full symmetry group of eleven-dimensional supergravity itself is. And furthermore as to why the exceptional groups emerge naturally in this context, as they had not appeared in this way in a physical theory before [67].[8]

In 1985 Bernard de Wit and Hermann Nicolai were able to prove that the local $\mathrm{SO}(1,10)$ symmetry of eleven-dimensional supergravity can be manifestly extended to a local $\mathrm{SO}(1,3)\times\mathrm{SU}(8)$ by combining degrees of freedom from the metric and the three-form [20, 21, 72]. Hermann Nicolai furthermore showed that the transformation rules and the bosonic field representations can be made manifestly SO(16) invariant [22].[9] As we will discuss in chapter 3 further progress was later made in enlarging the manifest symmetries of eleven-dimensional supergravity by reformulating the theory on generalised and extended notions of geometry.

We can write the lower dimensional Lagrangian in a form that is manifestly invariant under the full exceptional symmetry [1, 2, 66, 67, 74]. The vector fields are abelian and transform linearly under the global $\mathrm{E}_{n(n)}$, whereas the spinor fields transform linearly under the local $\mathrm{K}(\mathrm{E}_{\mathrm{n(n)}})$ [67]. The scalar fields are described by a non-linear $\mathrm{E}_{\mathrm{n(n)}}/\mathrm{K}(\mathrm{E}_{\mathrm{n(n)}})$ coset sigma model and transform under both groups [67]. In section 2.3.3 we discuss how to explicitly construct (ungauged) supergravity Lagrangians for a general coset symmetry. In chapter 5 we carry out a detailed canonical analysis for the manifestly $\mathrm{E}_{6(6)}$ invariant theory in five dimensions, which was first described in the Lagrangian formulation in [74].

The appearance of hidden (exceptional) symmetries in toroidal compactifications of eleven-dimensional supergravity can furthermore be compared to the symmetry enhancements that arise in the toroidal compactification of four-dimensional general relativity. The reduction of general relativity to three dimensions leads to the SL(2) Ehlers symmetry [75] and reduction to two dimensions leads to the infinite dimensional Geroch symmetry [76, 77] (which is the affine extension of the loop group $\widehat{\mathrm{SL}(2)}$ [69]).

[8]The related compact exceptional group E_6 has already been considered as a grand unified theory (GUT) symmetry candidate in 1975, however the symmetry did not emerge from the framework itself, as is the case in supergravity [71].

[9]See [73] for a review of [20–22, 72].

2.3.3 Group invariant Lagrangians

In general we can write the scalar part of the Lagrangian of ungauged supergravity as in equation (2.19), where the target space metric $G_{ij}(\phi)$ of the non-linear sigma model depends on the scalar fields ϕ^i and just like the scalar potential $V(\phi)$ it is constrained by the amount of supersymmetry of the theory [15, 48, 57].[10,11]

$$\mathcal{L}_{\text{sc.}} = -\frac{E}{2}\partial_\mu\phi^i\,\partial^\mu\phi^j\,G_{ij}(\phi) - E\,V(\phi) \tag{2.19}$$

For more than eight real supercharges the geometry of the target space is given by a symmetric space G/K (where G is a Lie group and K its maximal compact subgroup) [15, 48, 57]. However it is only in the case of maximal supersymmetry that this space is uniquely given by the cosets $\mathrm{E}_{n(n)}/\mathrm{K}(\mathrm{E}_{n(n)})$ — in this case the scalar potential has to vanish [15, 57].

Instead of this geometric picture we can describe the scalar fields in an algebraic way using the group elements $\mathcal{V}(\phi)\in G$, which are analogous to the vielbein of the Lorentz symmetry. The $\mathcal{V}$ transform as in equation (2.20) under global (i.e. constant) $\Lambda\in\mathfrak{g}$ and local $k(x)\in\mathfrak{k}$ transformations, where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{k}$ is the Lie algebra of $K(G)$ [57, 67].

$$\delta\mathcal{V} = \Lambda\,\mathcal{V} - \mathcal{V}\,k(x) \tag{2.20}$$

The Lie algebra $\mathfrak{g}$ of G can be written in terms of its Cartan decomposition (cf. Iwasawa decomposition) $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement (with respect to the Killing form) and the following Lie bracket relations hold for these subalgebras $[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subseteq\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{k}$ [79]. Using this we can define the left invariant currents $J_\mu\in\mathfrak{g}$ (2.21), with the compact $Q_\mu\in\mathfrak{k}$ and non-compact $P_\mu\in\mathfrak{p}$ parts [57, 67].

$$J_\mu := \mathcal{V}^{-1}\,\partial_\mu\mathcal{V} =: Q_\mu + P_\mu \tag{2.21}$$

We can use this to write the kinetic Lagrangian (2.19) as (2.22) [57, 67].

$$\mathcal{L}_{\text{sc.}} = -\frac{1}{2}\,E\,\mathrm{Tr}(P_\mu\,P^\mu) \tag{2.22}$$

In this picture Q_μ transforms under the local symmetry as a (composite) gauge connection $\delta Q_\mu = -\partial_\mu k + [k, Q_\mu]$ and P_μ as $\delta P_\mu = [k, P_\mu]$ [57, 67]. Furthermore if we were to consider fermions the Q_μ appear as the K-connection in their covariant derivatives, this is in analogy to the spin connection for the Lorentz symmetry. Using the Cartan involution θ (a Lie group G-automorphism with $\theta^2=\mathrm{id}_G$) we can furthermore define the analogue of the Lorentzian metric (2.23) which is manifestly K-invariant $\delta\mathcal{V}=\mathcal{V}\,k$ [57, 67].[12]

$$M := \mathcal{V}\,\theta(\mathcal{V})^{-1} \tag{2.23}$$

Using this scalar metric (2.23) we can again rewrite the scalar Lagrangian as (2.24) [57, 67].

$$\mathcal{L}_{\text{sc}} = +\frac{E}{8}\,\mathrm{Tr}(\partial_\mu M\,\partial^\mu M^{-1}) \tag{2.24}$$

[10] This section is mainly based on the lectures [48, 57].

[11] Most of the contents of this section can also be found in reference [78].

[12] For the simple case of the coset $\mathrm{SL}(n)/\mathrm{SO}(n)$ M is just $M=\mathcal{V}\,\mathcal{V}^T$ and it is obviously invariant under orthogonal transformations.

The p-form gauge fields A^M transform as $\delta_\Lambda A^M_{\mu_1\ldots\mu_p} = \Lambda_N{}^M A^N_{\mu_1\ldots\mu_p}$, with $\Lambda \in G$, in a linear representation under G. The scalar metric M_{MN} from (2.23) furthermore contracts the field strengths $F^M := \mathrm{d}A^M$ of the gauge fields in the p-form kinetic term (2.25) [57].

$$\mathcal{L}_{\text{p-kin}} \sim E\, F^M_{\mu_1\ldots\mu_{p+1}}\, F^{\mu_1\ldots\mu_{p+1}N}\, M_{MN} \tag{2.25}$$

An additional metric-independent topological (Chern-Simons) term, whose form depends on the dimension, can also exist. The G-representation of the p-form gauge fields needs to admit a (constant) group invariant symbol with the right number of indices to contract the p-form gauge fields and field strengths in the topological term. For example in five dimensions the topological term can take the form (2.26) if the invariant symbols d_{MNK} exist [15, 57, 74]. Because the field strengths are gauge invariant the topological term transforms as a total derivative if the group invariant symbol is constant.

$$\mathcal{L}_{\text{top}} \sim F^M \wedge F^N \wedge A^K\, d_{MNK} \tag{2.26}$$

Furthermore there exists an equivalent Hodge-dual description of the p-forms in d dimensions in terms of $(d-p-2)$-forms B_M by dualising the field strengths (see e.g. [48]). In the following the "$\star$" indicates Hodge dualisation.

$$G_M := \star M_{MN} F^N \tag{2.27}$$

We can define the Hodge-dual field strength G_M by (2.27) and locally we can define the $(d-p-2)$-forms B_M by $G_M =: \mathrm{d}B_M$. The Bianchi identity $\mathrm{d}F^M = 0$ and the dual Bianchi identity $\mathrm{d}G_M = 0$ hold and we can can write the (non-topological) equation of motion $\mathrm{d}\star(M_{MN}F^M) = 0$ equivalently as $\mathrm{d}(\star(M^{-1})^{MN}\, G_N) = 0$. Due to Hodge-duality there exist many on-shell equivalent descriptions of the theory. The description where all p-forms are dualised to the lowest possible degree is the formulation in which the exceptional symmetries can be displayed manifestly — we will come back to this point in chapter 3.

2.3.4 Gauged supergravity

As was mentioned in section 2.3.2 the geometry on which one carries out the compactification can be more general than an n-torus. For example a consistent reduction on a seven-sphere S^7 is also possible and leads to a gauging of the (S^7 isometry) group SO(8) in $d = 4$, $\mathcal{N} = 8$ supergravity [63]. The reduction on a four-sphere has also been shown to be consistent [64]. However not all geometries lead to consistent theories upon reduction. There are alternative ways of gauging a subgroup $G_0 < G$ of the global symmetry group directly in d-dimensional supergravity [78, 80–88].[13] This section is based on [48, 57].

In principle one can try to gauge any subgroup $G_0 < G$ of the global symmetry group, however the dimension of G_0 needs to be less or equal to the number of vector fields because we cannot introduce other gauge connections due to the supersymmetry constraints imposed on the field content of the theory. The generators X_M of the subgroup G_0 can then be expressed in terms of the generators t_ζ of G by $X_M = \Theta_M^{\zeta}\, t_\zeta$, where Θ_A^{ζ} is the embedding tensor which acts as a projector that encodes the local gauge symmetry. The embedding tensor is subject to consistency conditions coming

[13] A comprehensive introduction to the embedding tensor formalism for gauged supergravity can be found in [78].

from the group structure (called the quadratic constraints) and from supersymmetry (called the linear constraints). These consistency conditions restrict the choices of groups G_0 that lead to consistent gaugings. In particular the generators of the subgroup have to close under the bracket (2.28).

$$[X_M, X_N] = -X_{MN}{}^K X_K, \text{ with } X_{MN}{}^K := \Theta_M^\zeta\, (t_\zeta)_N{}^K \tag{2.28}$$

The vector fields A_μ^M transform initially linearly under G and under abelian gauge transformations. One can then write down covariant derivatives, which can be expressed as $D_\mu = \partial_\mu - g\, A_\mu^M\, \Theta_M^\zeta\, t_\zeta$, however there is an issue with this construction because the "structure constants" $X_{MN}{}^K$ are not anti-symmetric in general and we can define their symmetric and anti-symmetric parts as in (2.29).

$$X_{MN}{}^K =: -f_{[MN]}{}^K + Z_{(MN)}{}^K \tag{2.29}$$

The one-form field strength defined by (2.29) is then given by (2.30).

$$F_{\mu\nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M - A_\mu^K A_\nu^L\, f_{KL}{}^M \tag{2.30}$$

The $f_{[MN]}{}^K$ do not satisfy the Jacobi identity and have a non-vanishing Jacobiator (2.31).

$$f_{MN}{}^Q\, f_{KQ}{}^P + \text{cycl.} = -Z_{Q[M}{}^P\, f_{NK]}{}^Q \tag{2.31}$$

Equation (2.31) leads to an additional $Z_{(KL)}{}^M$ term in the transformation of $F_{\mu\nu}^M$ (2.32) and it does not transform covariantly.

$$\delta_\Lambda F_{\mu\nu}^M = -\Lambda^P\, X_{PN}{}^M\, F_{\mu\nu}^N + Z_{KL}{}^M \left(\Lambda^K\, F_{\mu\nu}^L - A_{[\mu}^K\, \delta A_{\nu]}^L\right) \tag{2.32}$$

A possible solution to this problem is the introduction of two-forms $B_{\mu\nu}^{(KL)}$ that are added to the one-form field strength as a Stückelberg coupling (2.33).

$$\mathcal{F}_{\mu\nu}^M := \partial_\mu A_\nu^M - \partial_\nu A_\mu^M - A_\mu^K A_\nu^L\, f_{KL}{}^M + Z_{KL}{}^M\, B_{\mu\nu}^{(KL)} \tag{2.33}$$

In some dimensions two-forms that can be used for this procedure are already present in the field content, but in other cases one has to dualise other fields into the two-forms. If the two-forms are introduced with a topological kinetic term no propagating degrees of freedom are added and they can be made dual to existing p-forms.
The gauge transformations of the p-forms can then be defined as (2.34) and (2.35) in order to make the two-forms absorb the non-covariant transformation from (2.32). Because of the introduction of the two-forms the fields now also transform under the two-form gauge transformations with parameter Ξ_ν^{KL}.

$$\delta A_\mu^M = D_\mu \Lambda^M - Z_{KL}{}^M\, \Xi_\mu^{KL} \tag{2.34}$$

$$\delta B_{\mu\nu}^{KL} = -\Lambda^{(K} \mathcal{F}_{\mu\nu}^{L)} + A_{[\mu}^{(K}\, \delta A_{\nu]}^{L)} + 2\, D_{[\mu} \Xi_{\nu]}^{KL} \tag{2.35}$$

The transformations (2.34) and (2.35) make the one-form field strength transform covariantly, but because we have introduced the two-forms they also come with a field strength $H_{\mu\nu\rho}^{KL}$ which does not transform covariantly. One can then proceed by introducing $(p+1)$-forms to repair the non-covariance, only to discover that the introduction of ever higher degree p-forms is required. This is known as the tensor hierarchy [78, 82–87]. Fortunately not all higher forms need to appear explicitly in the Lagrangian, e.g. in $d = 4, 5$ already the three-form does not appear in the

Lagrangian.
The full gauged supergravity should then be written in terms of the covariantised derivatives and field strengths, obeying the group specific consistency conditions on the embedding tensor that follow from supersymmetry. We will not need the full construction of gauged supergravity in this thesis, but it is useful to have the above construction in mind when we discuss exceptional field theory, which has a similar structure, in chapter 3.

2.4 Low energy effective action of string theory and string dualities

There is another perspective for thinking about the exceptional symmetries of supergravity, which were discussed in section 2.3.2, which is the perspective from the point of string theory. In this section we take a very brief look at string theory, how it relates to supergravity and how the exceptional symmetries arise in this context.

String theory is a framework of several different theories, with the fundamental idea being the extension of the action of a relativistic point particle to a one-dimensional extended string.[14] The Polyakov action of a bosonic string in a (26-dimensional) curved background, with background metric $G_{\mu\nu}$ can be written as the non-linear sigma model (2.36) [89].

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g} \, g^{\alpha\beta} \, \partial_\alpha X^\mu \, \partial_\beta X^\nu \, G_{\mu\nu}(X) \tag{2.36}$$

The generalisation of the world-line is the two-dimensional world-sheet of the string (with coordinates σ^α) and $g_{\alpha\beta}$ is an auxiliary world-sheet metric. From the world-sheet perspective the fields X^μ are scalar fields on the world-sheet. α' is the square of the string length scale.
The action (2.36) is invariant under world-sheet diffeomorphisms and in particular it is Weyl invariant, with the world-sheet metric transforming as $g_{\alpha\beta}(\sigma) \to \Omega(\sigma)\, g_{\alpha\beta}(\sigma)$ under Weyl transformations [89]. In the quantised bosonic string one can expand around a classical solution and find that the Weyl invariance requires the background metric $G_{\mu\nu}$ to be Ricci-flat $R_{\mu\nu} = 0$ [93]. The bosonic string theory (2.36) is therefore at low energies effectively described by general relativity — this is the low energy effective action. Even for bosonic string theory one can look at a more general background that also has a two-form $B_{\mu\nu}$ (Kalb-Ramond field or B-field) and a scalar field ϕ (dilaton). The consistency conditions of such a background can also be interpreted as equations of motion and add a two-form kinetic term $H_{\mu\nu\rho}\, H^{\mu\nu\rho}$ and a scalar kinetic term $\partial_\mu\phi\, \partial^\mu\phi$ to the low energy effective action of the (critical) bosonic string theory (2.37) [89, 91, 94, 95].

$$S_{\text{Eff.}} = \frac{1}{2\,\kappa_0^2} \int d^{26}x \, \sqrt{-G} \, e^{-2\phi} \left(R(G_{\mu\nu}) - \frac{1}{12}\, H_{\mu\nu\rho}\, H^{\mu\nu\rho} + 4\, \partial_\mu\phi\, \partial^\mu\phi \right) \tag{2.37}$$

In (2.37) $R(G_{\mu\nu})$ is the Ricci scalar, κ_0 is a constant, $H_{\mu\nu\rho}$ is the field strength of the two-form. For the supersymmetric string theories one finds similarly that the

[14] A textbook introduction to both the bosonic and superstring is given in the books [89, 90]. The Part III lecture notes by Prof. David Tong give a didactic introduction to the bosonic string [91]. The Part III lecture notes by Prof. Paul Townsend discuss the canonical formulation of string theory [92].

low energy effective actions are in fact given by supergravity actions [90]. This close relationship between supergravity and string theory makes supergravity an important tool for string theory calculations, but likewise string theory can serve as a different perspective on results in supergravity.

There is a web of duality symmetries between the different quantised superstring theories that relate solutions of one theory to those of another [89, 90]. T-duality relates solutions whose geometry includes an n-torus by exchanging the discrete momentum modes (regarding the compact hypertorus) with the winding number (i.e. winding of the string around the hypertorus) while simultaneously inverting the radii of the torus $R_i \to 1/R_i$ [89]. This toroidal duality can be described in terms of the discrete orthogonal group $\mathrm{O}(n,n;\mathbb{Z})$ [89]. S-duality (or strong-weak duality) relates strongly coupled theories to weakly coupled ones by inverting the string coupling $g_s \to 1/g_s$ [90]. Combining all duality transformations the unified or U-duality group $\mathrm{E}_{n(n)}(\mathbb{Z})$ is generated [90, 96]. It has been conjectured that there exists an eleven-dimensional theory, called M-theory, from which all the ten-dimensional superstring theories originate and for which U-duality would be a true symmetry [90, 97]. Furthermore the unique eleven-dimensional supergravity theory (cf. section 2.2.1) is thought to be the low energy effective action of this string M-theory. Therefore, from the string theory perspective, the origin of the exceptional symmetries $\mathrm{E}_{n(n)}(\mathbb{R})$, that arise from the toroidal compactifications of eleven-dimensional supergravity (cf. section 2.3.2), can be interpreted as being inherited from the U-duality symmetry of M-theory [96]. However in string theory the exceptional symmetry is discrete $\mathrm{E}_{n(n)}(\mathbb{Z})$ and for $n \leq 7$ this can be understood to be due to the Dirac quantisation of charges [96, 98]. This interpretation does not explain how much of the exceptional symmetry is present already in eleven-dimensional supergravity prior to any reduction. It does however give a possible interpretation for the existence of exceptional symmetries in supergravity in general, at least from the string theory perspective. Moreover it is a remarkable fact that any trace of the non-perturbative "stringy" duality symmetries would survive in the low energy field theory limit.

Chapter 3

Geometrisation and generalised notions of geometry

Geometrisation is the idea that seemingly unrelated (non-geometric) degrees of freedom of a theory can be described in terms of a common higher dimensional geometric origin. In some cases the gauge transformations of these degrees of freedom can be combined into generalised notions of diffeomorphisms.

In section 3.1 we look at Kaluza–Klein theory to illustrate the concept of geometrisation by describing how four-dimensional gravity coupled to Maxwell theory and a scalar field can be seen as originating from pure five-dimensional gravity. In section 3.2 we introduce generalised (complex) geometry, which generalises complex geometry by means of a broader notion of the tangent bundle. We briefly discuss how the same ideas can be applied to exceptional groups. In section 3.3 we introduce doubled geometry, which is an extension of generalised geometry, in which not only the tangent bundle but also the coordinates of the base manifold are extended (but we also comment on the more general construction of extended generalised geometries for general Lie groups). Furthermore we discuss double field theory, which is an $\mathrm{O}(n,n)$ covariant field theory built on doubled geometry, that encodes the string theory low energy effective action in a manifestly $\mathrm{O}(n,n)$ invariant form. The extended generalised geometry can be constructed for different Lie groups and in section 3.4 we consider the extended generalised symplectic geometry as an example and construct the Y-tensor explicitly. In section 3.5 we discuss the extended generalised exceptional geometry in detail, taking the $\mathrm{E}_{6(6)}$ case as an example. We then discuss the Lagrangian formulation of the $\mathrm{E}_{6(6)}$ exceptional field theory, which encodes the dynamics of eleven-dimensional supergravity in a manifestly $\mathrm{E}_{6(6)}$ covariant formulation based on an extended generalised exceptional geometry.

3.1 Basic idea: Geometrisation and Kaluza–Klein theory

Kaluza-Klein theory, developed in the 1920s by Theodor Kaluza and Oskar Klein, is the historical origin of the idea of geometrisation [23, 24].[1] Kaluza-Klein theory demonstrates that five-dimensional gravity can be understood as giving rise to four-dimensional gravity coupled to Maxwell theory and a massless scalar field upon compactification on a circle. The circular compactification of a scalar field was already explained in section 2.3.1 and the compactification of the five-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x}^{\hat{\rho}})$ proceeds in the same way — in fact the idea of compactification was originally popularised by Kaluza-Klein theory. In this section we follow the description of

[1] See references [62, 99–103] for modern discussions of Kaluza-Klein theory.

Kaluza-Klein theory presented in [103].

The Einstein-Hilbert action in five dimensions is given by (3.1) where $\hat{R}_5$ is the five-dimensional Ricci scalar and $\hat{\mu}, \hat{\nu}$ are five-dimensional space-time indices.

$$S_5 = \int d^5\hat{x}\, \sqrt{-\hat{g}}\, \hat{R}_5(\hat{g}_{\hat{\mu}\hat{\nu}}) \tag{3.1}$$

The action (3.1) is invariant under local diffeomorphisms. Infinitesimally the metric transforms under diffeomorphisms as its Lie derivative and in local coordinates we can write this transformation as (3.2). The first term in the Lie derivative (3.2) can be interpreted as a local translation and the second term as a local $\partial_{\hat{\mu}}\hat{\xi}^{\hat{\rho}} \in \mathrm{gl}(5)$ rotation.

$$\delta_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = L_{\hat{\xi}}\hat{g}_{\hat{\mu}\hat{\nu}} = \underbrace{\hat{\xi}^{\hat{\rho}}\, \partial_{\hat{\rho}}\hat{g}_{\hat{\mu}\hat{\nu}}}_{\text{translation}} + \underbrace{2\, \partial_{(\hat{\mu}}\hat{\xi}^{\hat{\rho}}\, \hat{g}_{\hat{\nu})\hat{\rho}}}_{\text{rotation}} \tag{3.2}$$

In order to compactify the theory, given by the action (3.1), on a circle (with a radius proportional to R and with a circular coordinate y) we Fourier expand the five-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x}^{\hat{\rho}})$ as (3.3), which is in direct analogy to (2.14).

$$\hat{g}_{\hat{\mu}\hat{\nu}}(\hat{x}^{\hat{\rho}}) = \sum_{n\in\mathbb{Z}} g^{(n)}_{\hat{\mu}\hat{\nu}}(x^\rho)\, e^{\frac{iny}{R}} \tag{3.3}$$

The indices μ, ν, ρ in (3.3) are four-dimensional. The $g^{(n)}_{\mu\nu}(x^\rho)$ are the Fourier modes, labelled by the mode number $n \in \mathbb{Z}$, that only depend on the (non-circular) four-dimensional coordinates x^ρ. Just like in section 2.3.1, only the massless zero-mode $g^{(0)}_{\mu\nu}(x^\rho)$ will survive after the compactification. The compactification of the circle can again be understood as applying the cylinder or Kaluza-Klein condition (2.18) which singles out the zero-mode. For a circular (or toroidal) compactification the truncation to the massless zero-mode is consistent because the massive modes are not sourced by the massless mode and hence decouple. Equivalently we can think of the zero-mode as the only singlet of the isometry group U(1) of the circle and the truncation to the singlet mode is consistent because it cannot generate any non-singlet modes, cf. [62, 104].

In contrast to a scalar field, the metric has tensorial indices which also need to be decomposed in a 4+1 Kaluza-Klein split $\hat{\mu} = (\mu, y)$. Naively we can try to decompose the metric as in (3.4) in terms of the four-dimensional metric $g_{\mu\nu}$, a one-form field A_μ and a scalar field ϕ.

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\nu & \phi \end{pmatrix} \tag{3.4}$$

While (3.4) is a valid way of carrying out this decomposition it turns out that the parametrisation (3.5) is a better way to arrive at the result we want.

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \phi^{-\frac{1}{2}}\, g_{\mu\nu} + \phi\, A_\mu A_\nu & \phi\, A_\mu \\ \phi\, A_\nu & \phi \end{pmatrix} \tag{3.5}$$

With the parametrisation (3.5) the diffeomorphism transformations (3.2) decompose into the following infinitesimal transformations of the four-dimensional fields [103]. The metric (3.6) and the scalar field (3.8) simply transform as their four-dimensional Lie derivatives. The one-form field transforms as the Lie derivative, but there is an additional U(1) gauge transformation term in (3.7) with gauge parameter Λ.

$$\delta_\xi g_{\mu\nu} = L_\xi g_{\mu\nu} = \xi^\rho\,\partial_\rho g_{\mu\nu} + 2\,\partial_{(\mu}\xi^\rho\, g_{\nu)\rho} \tag{3.6}$$

$$\delta_{\xi,\Lambda} A_\mu = L_\xi A_\mu + \partial_\mu \Lambda \tag{3.7}$$

$$\delta_\xi \phi = L_\xi \phi \tag{3.8}$$

Decomposing the five-dimensional Einstein-Hilbert action (3.1) and applying the Kaluza-Klein condition we find the four-dimensional action (3.9) [103]. In (3.9) we introduced the dilaton $\varphi := \frac{\sqrt{3}}{2}\ln(\phi)$ as a field redefinition of the scalar field ϕ, the abelian one-form field strength $F_{\mu\nu} := 2\,\partial_{[\mu}A_{\nu]}$ and the four-dimensional Ricci scalar R_4.

$$S_4 = \int d^4x\,\sqrt{-g}\left(R_4(g_{\mu\nu}) - \frac{e^{\sqrt{3}\varphi}}{4}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi\right) \tag{3.9}$$

The four-dimensional action (3.9) describes four-dimensional gravity coupled to Maxwell theory (Einstein-Maxwell theory) and coupled to the kinetic term of the massless scalar field φ. The coupling constants are fixed and the action is unique up to overall rescalings and field redefinitions. The action (3.9) is invariant under the transformations (3.6), (3.7) and (3.8).

We can think of the action (3.9) as arising from five-dimensional gravity through circular compactification, but conversely we can think of five-dimensional gravity (3.1) as being a geometrised description of four-dimensional gravity coupled to this precise set of further "non-geometric" degrees of freedom. Importantly for this geometrisation to be equivalent to the lower dimensional theory, we have to apply the Kaluza-Klein condition $\partial_y = 0$ to the higher-dimensional geometric description, which implies that some of the higher-dimensional degrees of freedom that become massive in the compactification are truncated. We can moreover see that the four-dimensional diffeomorphisms are unified with the U(1) gauge transformations to appear as five-dimensional diffeomorphisms.

In the following sections we look at generalised notions of geometry and how they can be used to geometrise the description of supergravity, which makes it possible to manifest larger (duality) symmetries. While there are some important differences between Kaluza-Klein theory and the theories that we discuss in the following sections, the idea of geometrisation is very much the same.[2] The main advantage of geometrisation is that it allows the display of a greater amount of symmetry and the use of geometric methods.

3.2 Generalised (complex) geometry

Generalised (complex) geometry was first introduced in 2002 by Nigel Hitchin in [106] and further developed by Marco Gualtieri [107] and Gil Cavalcanti.[3,4] In this section we follow the reference [108]. Generalised geometry generalises the notion of both symplectic and complex geometry by replacing the tangent bundle $\mathrm{T}(\mathcal{M})$ of the manifold $\mathcal{M}$ with an extended bundle (3.10) that is the sum of the tangent and

[2] This analogy has been emphasised e.g. in [102, 103, 105]

[3] For lecture notes on generalised geometry by Hitchin and Cavalcanti see [108, 109] respectively.

[4] Generalised complex geometry can also be seen as an extension of the concept of Dirac structures and Dirac manifolds [110].

cotangent bundles.

$$\mathrm{E} := \mathrm{T}(\mathcal{M}) \oplus \mathrm{T}^*(\mathcal{M}) \tag{3.10}$$

The structure group of the extended bundle is the orthogonal group $\mathrm{O}(n,n)$, with $n = \dim(\mathcal{M})$. The Lie bracket is replaced by the Courant bracket (3.11) where $X, Y \in \mathrm{T}(\mathcal{M})$ and $\xi, \lambda \in \mathrm{T}^*(\mathcal{M})$, which does however not satisfy the Jacobi identity.

$$[X+\xi, Y+\lambda] := [X,Y] + \mathcal{L}_X\lambda - \mathcal{L}_Y\xi - \frac{1}{2}\mathrm{d}(\lambda(X) - \xi(Y)) \tag{3.11}$$

Furthermore there exists a natural (indefinite) inner product (3.12) that can be considered as being the analogue of the Minkowski metric, i.e. an $\mathrm{O}(n,n)$ metric (3.13).

$$\eta(X+\xi, Y+\lambda) := \frac{1}{2}(\xi(Y) + \lambda(X)) \tag{3.12}$$

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.13}$$

There is a naturally appearing two-form $B \in \Lambda^2\mathrm{T}^*(\mathcal{M})$, which acts on $X+\xi \in \mathrm{T}\oplus\mathrm{T}^*$ as (3.14), where i is the interior product.

$$B: \; X+\xi \to X+\xi+i_X B \tag{3.14}$$

The inner product and the Courant bracket are invariant under diffeomorphisms and under the action of B if the two-form is closed. The local diffeomorphism invariance under $\mathrm{Diff}(\mathcal{M})$ is then extended to the group (3.15), which is the semi-direct product of closed two-forms on $\mathcal{M}$ with the diffeomorphisms of $\mathcal{M}$.

$$\Omega^2_{\mathrm{cl.}}(\mathcal{M}) \rtimes \mathrm{Diff}(\mathcal{M}) \tag{3.15}$$

The Lie algebra associated to the Lie group (3.15) consists of the sections of $\mathrm{T} \oplus \Lambda^2\mathrm{T}$ and the antisymmetrised action of these sections on the sections of $\mathrm{T} \oplus \mathrm{T}^*$ is precisely given by the Courant bracket (3.11). In generalised (complex) geometry the diffeomorphisms and tensor gauge transformations are hence treated as being on the same footing, i.e. the tensor gauge transformations are geometrised in this framework. Therefore one can define a generalised Lie derivative (Dorfman derivative) that encodes not just the original diffeomorphisms but also the gauge transformations of the two-form [108, 111–114]. For $\varphi \in \Lambda^*\mathrm{T}^*$ and $u = X+\xi \in \mathrm{T}\oplus\mathrm{T}^*$ we can define the product (3.16) and using this we can define the generalised Lie derivative (3.17) along the direction of the section u.

$$(X+\xi)\cdot\varphi := \varphi(X) + \xi\wedge\varphi \tag{3.16}$$

$$\mathbb{L}_u\varphi := \mathrm{d}(u\cdot\varphi) + u\cdot\mathrm{d}\varphi \tag{3.17}$$

For specific types of spacetimes (that are warped products of lower dimensional manifolds with Minkowski space) one can rewrite eleven-dimensional supergravity as generalised gravity [70, 113, 114].
It is important to note that generalised (complex) geometry only extends the tangent bundle but leaves the dimension of the base manifold $\mathcal{M}$ unchanged.

3.2.1 Generalised exceptional geometry

Motivated by the existence of hidden symmetries in eleven-dimensional supergravity [21, 22, 115] one can extend the idea of generalised (complex) geometry to even larger bundles, such as (3.18), in order to accommodate exceptional structure groups $\mathrm{E}_{n(n)}$ ($n \leq 7$) in such a generalised exceptional geometry and this was done in [70, 114, 116, 117].

$$\mathrm{E} := \mathrm{T}(\mathcal{M}) \oplus \Lambda^2\mathrm{T}^*(\mathcal{M}) \oplus \Lambda^5\mathrm{T}^*(\mathcal{M}) \oplus (\mathrm{T}^*(\mathcal{M}) \otimes \Lambda^7\mathrm{T}^*(\mathcal{M})) \tag{3.18}$$

The terms in (3.18) can be interpreted physically as various brane charges, i.e. the momentum, M2- and M5-brane charges and the Kaluza-Klein Monopole charge [114, 117, 118].[5] The precise form of the bundle one should consider depends on the exceptional group, e.g. for $\mathrm{E}_{6(6)}$ only the additional two- and five-form terms are necessary (see table 2 in [70]).

Similar to generalised complex geometry one can rewrite eleven-dimensional supergravity in the generalised exceptional geometry for a certain class of space-times that have a warped Minkowski factor [114, 117]. For now we focus only on the geometry and do not worry too much about how it relates to any specific theory or physical aspects thereof.

The generalised Lie derivative (3.17) of generalised (exceptional) geometry, that encodes diffeomorphisms and p-form gauge transformations, can also be written as (3.19) in order to make the exceptional symmetry $\mathrm{E}_{n(n)}$ apparent [114, 117].

$$\mathbb{L}_U V^M = U^N \partial_N V^M - \alpha\, \mathbb{P}^M{}_N{}^K{}_L\, \partial_K U^L\, V^N + \lambda(V)\, \partial_N U^N\, V^M \tag{3.19}$$

The indices M, N, K, L are in a representation of the exceptional group $\mathrm{E}_{n(n)}$ (for many n this is the fundamental representation), the partial derivatives are identical to the coordinate derivatives of the manifold $\mathcal{M}$, i.e. $\partial_M = \partial_\mu$ and else vanishing i.e. $\partial_M = 0$. $\mathbb{P}^M{}_N{}^K{}_L$ is a projector onto the adjoint representation of $\mathrm{E}_{n(n)}$, $\lambda(V)$ is a generalised weight, $\alpha \in \mathbb{R}$ and U, V are sections of the extended tangent bundle (3.18).

The expression (3.19) seems somewhat strange given that we do not actually have as many coordinates as the derivative ∂_M seems to suggest. In the following sections we discuss extensions of generalised geometry that do not just extend the tangent bundle but also the underlying manifold, thus justifying the notation in (3.19).

3.3 Doubled geometry and double field theory (DFT)

In this section we look at doubled geometry to illustrate how generalised (complex) geometry can be further generalised by not just enlarging the tangent bundle but also extending the underlying manifold. We discuss this extended geometry first for the orthogonal $\mathrm{O}(n, n)$ structure group because this case is simpler than the exceptional cases and for now we will mainly focus on the geometric ideas. We briefly discuss how a doubled field theory (DFT) [29, 30, 119] can be constructed in terms of the doubled geometry, which is manifestly T-duality invariant and encodes the string theory low

[5] Branes are in general objects with two or more dimensions, however there is a range of very different ideas. Some are postulated as fundamental objects generalising the idea of fundamental particles and strings (p-branes), they can be described in terms of higher dimensional sigma-model "world-volume" actions. D-branes arise from boundary conditions of open strings. Black p-branes on the other hand can be thought of as higher dimensional black hole solutions, e.g. of supergravity. Similarly the M-branes mentioned here are eleven-dimensional supergravity solutions (black branes) that are charged with respect to some p-forms.

energy effective equations of motion (and hence also supergravity).[6]

Starting again with a n-dimensional manifold $\mathcal{M}$ we extend the tangent bundle to the bundle (3.10) from generalised (complex) geometry. The local coordinates on the manifold $\mathcal{M}$ are x^μ with $\mu = 1, \ldots, n$. Moreover we introduce "dual coordinates" $\tilde{x}_\mu$ and collectively write the coordinates as the O(n,n) vector X^M ($M = 1, \ldots, 2n$) of extended coordinates as in (3.20).

$$X^M = \begin{pmatrix} x^\mu \\ \tilde{x}_\mu \end{pmatrix} \tag{3.20}$$

Simultaneously we introduce the condition (3.21) on the coordinate derivatives associated to the extended coordinates (3.20), with η^{MN} being the inverse of the O(n,n) invariant (3.13).

$$\eta^{MN}\,\partial_M \otimes \partial_N = 0 \tag{3.21}$$

Equation (3.21) is called the section condition and it is interpreted to hold on all functions on the doubled geometry, with the derivatives either on the same or on different functions [123].

There is a lot going on in this construction and we should take a step back. We introduced the additional dual coordinates $\tilde{x}_\mu$ in order to reflect the extension of the tangent bundle (3.10) — however it is not clear that there exist ordinary manifolds that have such tangent bundles and indeed it seems to be the case that we have to extend the notion of manifold to allow for this construction globally [105, 122, 124–126]. As the global properties of this coordinate extended doubled geometry are not completely clear we generally work in local coordinates. The section condition (3.21) can be interpreted as a consistency condition and in particular it implies that only half of the $2n$ coordinates X^M actually exist [123]. Therefore the doubled geometry and the generalised complex geometry are locally equivalent [113].

Physically one can think of the ordinary coordinates x^μ as being associated to momentum modes, while the dual coordinates $\tilde{x}_\mu$ are associated to winding modes, in the sense of strings winding around compact toroidal dimensions [29, 30]. T-duality mixes these modes and therefore it makes sense physically to combine all of the coordinates in the O(n,n) vector X^M of the extended coordinates. This doubled geometry is what makes it possible to talk about the O(n,n) T-duality of string theory in terms of a field theory that would otherwise have no notion of T-duality. The section condition of doubled geometry (3.21) can (partially) be interpreted as coming from the level matching condition of string theory, which relates left and right moving modes of closed strings [123].

Because the explicit structure of extended generalised geometries varies significantly depending on the structure group one can introduce the Y-tensor $Y^{MQ}{}_{PR}$, which helps, to some extent, in unifying the description of extended generalised geometries. How the Y-tensor can be constructed for general Lie algebras was described in references [42, 43, 127] (although there is no guarantee that there exists a consistent extended generalised geometry for any Lie algebra).[7] In section 3.4 we construct

[6] See references [120–122] for reviews of double field theory.

[7] It is actually possible to be more general and one can construct an extended generalised geometry for any Kac-Moody algebra and any coordinate representation thereof, however closure of the algebra of the generalised diffeomorphisms is not guaranteed [42, 43].

the Y-tensor for the generalised Sp($2n$) symplectic geometry as an example. For the Riemannian structure group GL(n) the Y-tensor is vanishing and we can interpret the Y-tensor as describing corrections to Riemannian geometry [42]. The Y-tensor of O(n,n) doubled geometry is ${Y^{MN}}_{KL} = \eta^{MN}\,\eta_{KL}$ and the section condition can also be formulated in terms of the Y-tensor as (3.22). In the form (3.22) the section condition holds true for any structure group and due to the vanishing Y-tensor the section condition of Riemannian geometry is trivial [42, 117, 127].

$$ {Y^{MN}}_{KL}\,\partial_M \otimes \partial_N = 0 \tag{3.22} $$

We can write the O(n,n) version of the generalised Lie derivative (3.19) acting on a generalised vector, in local coordinates, as (3.23), with $\alpha = 2$ [103].

$$ \mathbb{L}_U V^M := \underbrace{U^N\,\partial_N V^M}_{\text{translation}} - \underbrace{\alpha\,{\mathbb{P}^M}_N{}^K{}_L\,\partial_K U^L\,V^N}_{\text{o}(n,n)\text{ rotation term}} + \underbrace{\lambda(V)\,\partial_N U^N\,V^M}_{\text{weight term}} \tag{3.23} $$

In contrast to generalised complex geometry the coordinate derivatives ∂_M all have corresponding extended coordinates — at least until we solve the section condition. Furthermore we allow a weight term, with $\lambda(V)$ being the generalised diffeomorphism weight of the generalised vector V^M. The definition (3.23) implies the transformation of general O(n,n) tensors by requiring the Leibniz property to hold. The projector ${\mathbb{P}^M}_N{}^K{}_L$ onto the adjoint representation of O(n,n) is explicitly given as (3.24) [103].

$$ \alpha\,{\mathbb{P}^M}_N{}^K{}_L := \delta^M_L\,\delta^K_N - \eta^{MK}\,\eta_{NL} \tag{3.24} $$

Compared to the normal gl(n) rotation term in the Riemannian Lie derivative we can interpret ${\mathbb{P}^M}_N{}^K{}_L\,\partial_K U^L$ as an o(n,n) transformation acting on the generalised vector V^N.

In general we can write the generalised Lie derivative in terms of the Y-tensor as (3.25), which makes it apparent that the Y-tensor represents a correction term to the standard Lie derivative [42, 43, 103, 127].

$$ \mathbb{L}_\Lambda V^M = \Lambda^N\,\partial_N V^M - V^N\,\partial_N \Lambda^M + {Y^{MQ}}_{PR}\,\partial_Q \Lambda^P\,V^R + (\lambda(V)+\omega)\,\partial_N U^N\,V^M \tag{3.25} $$

The coefficient α and the special weight ω are structure group dependent constants and for Riemannian and doubled geometry $\omega = 0$ [103]. The generalised Lie derivative can in general be written as both (3.23) or (3.25) if the correct constants and the appropriate Y-tensor or adjoint projector are inserted. The Y-tensor is closely related to the adjoint projector by (3.26) [103, 127].

$$ {Y^{MN}}_{PQ} = -\alpha\,{\mathbb{P}^M}_Q{}^N{}_P + {\delta^M}_P\,{\delta^N}_Q - \omega\,{\delta^M}_Q\,{\delta^N}_P \tag{3.26} $$

In doubled geometry one can check in particular that the generalised Lie derivative is compatible with the O(n,n) invariant η_{MN} (3.27).

$$ \mathbb{L}_U \eta_{MN} = 0 \tag{3.27} $$

In general the generalised Lie derivative should be compatible with the invariant tensors in the coordinate representation and accordingly with the Y-tensor. Using the section condition we find that generalised diffeomorphism parameters of the form (3.28), where T is some scalar function, lead to trivial generalised diffeomorphisms

$\mathbb{L}_U = 0$ and we refer to parameters of this form as trivial.

$$U^M = \eta^{MN}\,\partial_N T \tag{3.28}$$

Looking at the commutator of generalised Lie derivatives (3.29) we find that they close into the C-bracket defined as (3.30) if and only if we apply the section condition [128].

$$[\mathbb{L}_U, \mathbb{L}_V] = \mathbb{L}_{[U,V]_{\mathrm{C}}} \tag{3.29}$$

In this sense the section condition can be seen as a consistency condition that is required for the algebra of generalised diffeomorphisms to close.

$$[U,V]_{\mathrm{C}}^M = U^N\,\partial_N V^M - V^N\,\partial_N U^M - \frac{1}{2}\eta^{MN}\,\eta_{KL}(U^K\,\partial_N V^L - V^K\,\partial_N U^L) \tag{3.30}$$

The C-bracket is bilinear and anti-symmetric, however it fails to satisfy the Jacobi identity by a trivial term [128]. When the section condition is applied the C-bracket of doubled geometry (3.30) is equivalent to the Courant bracket of generalised geometry (3.11) [128].

The generalised bracket, which we may call $[[U,V]]$, can in general be written in terms of the Y-tensor as (3.31), which can be seen as a Y-tensor correction to the generalised commutator [127].

$$[[U,V]]^M = U^N\,\partial_N V^M - V^N\,\partial_N U^M - \frac{1}{2}Y^{MN}{}_{KL}(U^K\,\partial_N V^L - V^K\,\partial_N U^L) \tag{3.31}$$

In generalised geometry we saw that the diffeomorphisms and two-form gauge transformations are combined (3.15) and therefore the metric and two-form should be treated on the same footing. We can introduce a generalised metric $\mathcal{H}_{MN}(G,B)$ that parametrises the coset $\mathrm{O}(n,n)/(\mathrm{O}(n)\times\mathrm{O}(n))$ and therefore leaves the $\mathrm{O}(n,n)$ symbol η_{MN} invariant (3.32) [129].

$$\mathcal{H}_{MK}\,\eta^{KL}\,\mathcal{H}_{LN} = \eta_{MN} \tag{3.32}$$

We can express the generalised metric explicitly as (3.33) [129].[8] The generalised diffeomorphism transformations of (3.33) are, up to the section condition, equivalent to the standard diffeomorphisms and two-form gauge transformations of $G_{\mu\nu}$ and $B_{\mu\nu}$.

$$\mathcal{H}_{MN}(G,B) = \begin{pmatrix} G_{\mu\nu} - B_{\mu\rho}\,G^{\rho\sigma}\,B_{\sigma\nu} & B_{\mu\rho}\,G^{\rho\nu} \\ -G^{\mu\sigma}\,B_{\sigma\nu} & G^{\mu\nu} \end{pmatrix} \tag{3.33}$$

Concerning the construction of a field theory on this doubled geometry we need to introduce a (weight one) scalar field (generalised dilaton), which we call d, in order to write down an appropriate measure. For the fields $\mathcal{H}_{MN}$ and d one can then construct the unique action principle (3.34), which is manifestly invariant under $\mathrm{O}(n,n)$ generalised diffeomorphisms and moreover encodes the low energy effective action of either the bosonic string (with the critical dimension $n = 26$, cf. equation (2.37)) or the NSNS sector of the superstring (with the critical dimension $n = 10$) [30, 129, 131].

$$S_{\mathrm{DFT}} = \int d^{2n}X\,e^{-2d}\,\mathcal{R}_{\mathrm{DFT}}(\mathcal{H},d) \tag{3.34}$$

[8] This form of the generalised metric encodes the Buscher rules that describe T-duality transformations on these fields [130].

$\mathcal{R}_{\text{DFT}}$ is given explicitly by the function (3.35) and determined by requiring invariance under the generalised diffeomorphisms [129].

$$\begin{aligned}\mathcal{R}_{\text{DFT}}(\mathcal{H}, d) :=& \frac{1}{8}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_N\mathcal{H}_{PQ} - \frac{1}{2}\,\mathcal{H}^{MN}\,\partial_M\mathcal{H}^{PQ}\,\partial_P\mathcal{H}_{QN}\\ &+ 4\,\partial_M\mathcal{H}^{MN}\,\partial_N d - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d\\ &- \partial_M\partial_N\mathcal{H}^{MN} + 4\,\mathcal{H}^{MN}\,\partial_M\partial_N d \end{aligned} \tag{3.35}$$

The integral in (3.34) is taken to be symbolical since we do not actually know how to integrate on this doubled geometry while observing the section condition. To carry out the integral explicitly we need to solve the section condition, which breaks the $\mathrm{O}(n, n)$ covariance. One can think of (3.34) as a tool to encode the equations of motion in a symmetrical form, however we cannot associate a number to this action for a given solution of the theory. It is therefore not clear for example how one should approach the action (3.34) in the path integral formalism and hence one cannot quantise the theory this way. Recently the geometric quantisation of DFT has been discussed in [32].

Because T-duality in this formalism is simply an $\mathrm{O}(n, n)$ transformation and because (3.34) is manifestly $\mathrm{O}(n, n)$ invariant by construction, double field theory is manifestly T-duality invariant — i.e. its equations of motion are T-duality invariant.

The action (3.34) was analysed in the canonical formalism in reference [31].[9]

A manifestly T-duality invariant double sigma model, for which double field theory is the low energy effective field theory, was constructed in [132–134]. Its canonical formulation was worked out in [135].

3.3.1 Kaluza-Klein rewriting of DFT

It is possible to partially gauge fix the local Lorentz symmetry and rearrange the degrees of freedom described by the double field theory action (3.34) in a way that is in direct analogy to a Kaluza-Klein compactification — but without actually carrying out any truncation of degrees of freedom [136]. It is this formulation of double field theory that is best suited to be considered for the extension to the exceptional groups. We keep the description of this reformulation of DFT very brief and focus only on the main idea of rewriting the theory in a Kaluza-Klein-like split. In this section we follow reference [136].

The DFT space-time coordinates (3.20) can be split according to (3.36), where the index $\mu = 1, \ldots, n$ is split into what we may think of as external non-compact $\overline{\mu} = 0, \ldots, n - d - 1$ and internal (compact) $\Delta = 1, \ldots, d$ coordinates and their duals.

$$x^\mu =: (x^{\overline{\mu}}, y^\Delta) \qquad\qquad \tilde{x}_\mu =: (\tilde{x}_{\overline{\mu}}, \tilde{y}_\Delta) \tag{3.36}$$

By partially solving the section condition with $\tilde{\partial}^{\overline{\mu}} = 0$ we can drop the dual non-compact coordinates $\tilde{x}_{\overline{\mu}}$. The internal coordinates can be arranged as in equation

[9] In order for this to be possible it is assumed in [31] that nothing depends on the dual time coordinate. The reformulation of DFT described in section 3.3.1 does not have this difficulty — while a canonical analysis of this formulation has not been carried out so far the results should be equivalent because no truncation has been carried out in the rewriting.

(3.37) with $\overline{M} = 1, \ldots, 2d$.

$$Y^{\overline{M}} = (y^{\Delta}, \tilde{y}_{\Delta}) \tag{3.37}$$

The final set of coordinates is then $(x^{\overline{\mu}}, Y^{\overline{M}})$ where $x^{\overline{\mu}}$ are external non-compact and $Y^{\overline{M}}$ internal. In the remainder of this section we will cease to write the overline notation for these indices and write (x^{μ}, Y^{M}) where $\mu = 1, \ldots, \mathbf{n} = n - d - 1$ and $M = 1, \ldots, \mathbf{d} = 2d$.

This rewriting of DFT is similar to the Kaluza-Klein inspired rewritings of eleven-dimensional supergravity in [20, 22], however the theory still depends on the dual compact coordinates and the section condition applies.

The field content is given by (3.38).

$$\{G_{\mu\nu},\, B_{\mu\nu},\, \phi,\, \mathcal{H}_{MN},\, A_{\mu}^{M}\} \tag{3.38}$$

$G_{\mu\nu}$ and $B_{\mu\nu}$ are the external components of the metric and the two-form of DFT, the internal components are contained in the generalised metric $\mathcal{H}_{MN}$. The scalar field ϕ is a redefinition of the generalised dilation field. The vector fields A_{μ}^{M} are the Kaluza-Klein vector (cf. section 3.1).

The Kaluza-Klein vector fields A_{μ}^{M} are used as a gauge connection for the generalised diffeomorphisms and external covariant derivatives (3.39) are introduced.

$$\mathcal{D}_{\mu} := \partial_{\mu} - \mathbb{L}_{A_{\mu}} \tag{3.39}$$

In direct analogy to the tensor hierarchy of gauged supergravity (cf. section 2.3.4) the one-form field strength defined with the C-bracket fails to transform covariantly and an additional Stückelberg-type coupling to the two-form $B_{\mu\nu}$ has to be added (3.40).

$$\mathcal{F}_{\mu\nu}^{M} := 2\,\partial_{[\mu} A_{\nu]}^{M} - [A_{\mu}, A_{\nu}]_{\mathrm{C}}^{M} + \eta^{MK}\,\partial_{K} B_{\mu\nu} \tag{3.40}$$

The two-form field strength (3.41) likewise requires an additional Chern-Simons three-form contribution, but the tensor hierarchy terminates in this case and no three-form is needed as (3.41) is already covariant.

$$\mathcal{H}_{\mu\nu\rho} = 3\left(\mathcal{D}_{[\mu} B_{\nu\rho]} + \eta_{MN}\, A_{[\mu}^{N}\, \partial_{\nu} A_{\rho]}^{M} - \frac{1}{3}\eta_{MN}\, A_{\mu}^{N}\, [A_{\nu}, A_{\rho]}]_{\mathrm{C}}^{M}\right) \tag{3.41}$$

Using these covariant field strengths the DFT action can be rewritten as (3.42).

$$\begin{aligned} S_{\mathrm{DFT}} = \int d^{\mathbf{d}}Y \int d^{\mathbf{n}}x\, \sqrt{-G}\, e^{-2\phi} \Big(&\hat{R} + 4\, G^{\mu\nu}\, \mathcal{D}_{\mu}\phi\, \mathcal{D}_{\nu}\phi - \frac{1}{12}\mathcal{H}^{\mu\nu\rho}\, \mathcal{H}_{\mu\nu\rho} \\ &+ \frac{1}{8} G^{\mu\nu}\, \mathcal{D}_{\mu}\mathcal{H}^{MN}\, \mathcal{D}_{\nu}\mathcal{H}_{MN} - \frac{1}{4}\mathcal{H}_{MN}\, \mathcal{F}^{\mu\nu M}\, \mathcal{F}_{\mu\nu}^{N} - V \Big) \end{aligned} \tag{3.42}$$

The internal integral $\int d^{\mathbf{d}}Y$ is again understood to be symbolic, because the section condition needs to observed during the integration. $\hat{R}$ is the modified Ricci scalar of the external metric, it contains an additional one-form field strength term $+\mathcal{F}_{\alpha\beta}^{M}\, E^{\alpha\rho}\, \partial_{M} E_{\rho}{}^{\beta}$ where $E_{\rho}{}^{\beta}$ is the external vielbein. The scalar potential V is given by (3.43).

$$V(\phi, \mathcal{H}_{MN}, G_{\mu\nu}) = -\mathcal{R}(\phi, \mathcal{H}) - \frac{1}{4}\mathcal{H}^{MN}\,\partial_M G^{\mu\nu}\,\partial_N G_{\mu\nu} \tag{3.43}$$

The action (3.42) is manifestly $\mathrm{O}(\mathbf{d},\mathbf{d})$ invariant.

The Kaluza-Klein rewriting of the DFT action, which we reviewed in this section, is conceptually very useful, because the Kaluza-Klein-like construction of the $\mathrm{E}_{6(6)}$ exceptional field theory, which we discuss in detail in section 3.5, is in many ways very similar. The fully internal E_{11} exceptional field theory [137] may be thought of as the exceptional analogue to the original fully doubled DFT action — this analogy should become clearer after we have discussed the construction of exceptional field theory.

3.4 Excursion: Generalised Sp($2n$) symplectic geometry

In this section we make an excursion into generalised symplectic geometry — which is not normally considered because there does not seem to be any physical application for it so far. Nonetheless it is a nice example to contrast ordinary symplectic geometry with generalised symplectic geometry and moreover it illustrates how the Y-tensor can be constructed for a general group. In this section we follow the procedure described in the references [42, 43].[10]

If we look at the ordinary Lie derivative (3.44) of a vector V^μ then we find, using Darboux's theorem, that $\partial_\rho \xi^\mu \in sp(2n)$ if and only if ξ is a symplectic field, meaning $L_\xi \Omega = 0$, where Ω is a symplectic form (i.e. a closed, non-degenerate two-form) [139].

$$L_\xi V^\mu = \xi^\rho \partial_\rho V^\mu - V^\rho \partial_\rho \xi^\mu \tag{3.44}$$

In this case the Lie derivative (3.44) describes a symplectomorphism. For symplectic manifolds (i.e. manifolds that admit a symplectic form) one can consider symplectomorphisms as a subset of the standard diffeomorphisms with the additional condition that the parameter is a symplectic field. This is equivalent to the statement that the diffeomorphism preserves the symplectic form, i.e. the compatibility condition $L_\xi \Omega = 0$ holds. The rotation term of the generalised Lie derivative of generalised symplectic geometry is also an element of the $\mathrm{sp}(2n)$ algebra, however it turns out we get a very different kind of geometry. In particular we find a non-vanishing section condition, as opposed to the trivial vanishing section condition of $\mathrm{gl}(n)$, which is valid for both Riemannian and ordinary symplectic geometry.

In analogy to the approach taken in reference [42, 43] we now proceed to construct the Y-tensor of generalised symplectic geometry.
The symplectic algebra $\mathrm{sp}(2n)$ is defined by the condition (3.45) for any algebra element X and with Ω being the $\mathrm{sp}(2n)$ symplectic form.

$$X^T \Omega + \Omega X = 0 \tag{3.45}$$

We can define the inverse of the symplectic form by $\Omega^{kl}\Omega_{lm} = -\delta^k{}_m$, with the fundamental indices $k, l, m = 1, \dots, 2n$. Defining $X_{ml} := \Omega_{mk} X^k{}_l$ we can see that (3.45)

[10] See also the following references for more information about generalised diffeomorphisms [42, 43, 127, 138].

implies $X_{[ml]} = 0$ and therefore the generators of the algebra need to be symmetric. There are $(2n+1)n = \dim(\mathrm{sp}(2n))$ many generators of sp($2n$). Equation (3.46) is a general ansatz for the generators of sp($2n$) in the fundamental representation.

$$(T^r{}_s)^k{}_l = \alpha\, \delta^r{}_s\, \delta^k{}_l + \beta\, \delta^r{}_l\, \delta^k{}_s + \gamma\, \Omega^{rk}\, \Omega_{sl} \tag{3.46}$$

In the ansatz (3.46) the fundamental indices $k, l = 1, \ldots, 2n$ label the (fundamental) coordinate representation of sp($2n$), while the index pair $r, s = 1, \ldots, 2n$ labels the generators. The real coefficients α, β, γ of this ansatz are to be determined. The expression for the representation of the generators $(T^i{}_j)^k{}_l$ is required to satisfy the condition (3.47), which is equivalent to the definition (3.45).

$$(T^i{}_j)^r{}_s\, \Omega^{sk}\, \Omega_{rl} = -(T^i{}_j)^k{}_l \tag{3.47}$$

Solving for the coefficients α, β, γ we find that the generators in the fundamental representation are explicitly given by (3.48).

$$(T^r{}_s)^k{}_l = \delta^r{}_l \delta^k{}_s - \Omega^{rk} \Omega_{sl} \tag{3.48}$$

We can verify that this is indeed a representation of sp($2n$) by checking that the algebra (3.49) of these generators closes and we find that the structure constants are given by (3.50).

$$[T^k{}_l, T^r{}_s] = f^k{}_l{}^r{}_s{}^p{}_m\, T^m{}_p \tag{3.49}$$

$$f^k{}_l{}^r{}_s{}^p{}_m := \delta^k{}_s\, \delta^r{}_m\, \delta^p{}_l - \delta^p{}_l\, \Omega^{rk}\, \Omega_{sm} - \delta^r{}_m\, \Omega^{kp}\, \Omega_{ls} + \delta^r{}_l\, \Omega^{kp}\, \Omega_{sm} \tag{3.50}$$

The trace (3.51) of the representation indices defines an inner product and this expression is proportional to the Cartan-Killing form because it automatically satisfies the defining invariance condition. The factor $N^{-1} \in \mathbb{R}$ is a normalisation factor that will be determined below.

$$\kappa^k{}_l{}^r{}_s := \frac{1}{N}\, (T^k{}_l)^m{}_p\, (T^r{}_s)^p{}_m = \frac{2}{N} \left(\delta^k{}_s\, \delta^r{}_l - \Omega^{kr}\, \Omega_{ls} \right) \tag{3.51}$$

To find the inverse κ^{-1} of the inner product (3.51) we need to require that the inversion is relative to the identity that is of the same form as the expression of the generators (3.48), but with all indices being interpreted as labelling the generators. The inverse is found to be (3.52).

$$(\kappa^{-1})^k{}_l{}^r{}_s = \frac{N}{4} \left(\delta^k{}_s\, \delta^r{}_l - \Omega^{kr}\, \Omega_{ls} \right) \tag{3.52}$$

In order to calculate the Y-tensor we now need to compute some quantities determined by the representation theory of sp($2n$). We will carry out this computation explicitly for sp(4), but the results can be generalised to all sp($2n$). The Dynkin diagram of the symplectic algebra sp(4) is that of C_2 $\bullet\!\Leftarrow\!\bullet$ and there are two simple roots α_1 and α_2. From the Cartan matrix equivalent to the Dynkin diagram we can construct the root lattice and find that the set of positive roots is given by $\{\alpha_1, \alpha_2, (\alpha_1+\alpha_2), (2\alpha_1+\alpha_2)\}$. The fundamental weights of sp(4) are $w_1 = \alpha_1 + \frac{1}{2}\alpha_2$ and $w_2 = \alpha_1 + \alpha_2$. The highest weight of the four-dimensional fundamental representation of sp(4) is $\Lambda = w_1$ and the remaining weights of this representation are $\pm(w_1 - w_2)$ and $-w_1$. The norm of the highest weight Λ of the fundamental representation of any sp($2n$) is given by $(\Lambda, \Lambda) = \frac{1}{2}$. For sp(4) this implies that $1 = (\alpha_1, \alpha_1) = -(\alpha_1, \alpha_2) = \frac{1}{2}(\alpha_2, \alpha_2)$. The Weyl vector is defined by (3.53), which is the sum over all positive roots times 1/2

[140].

$$\varrho := \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha = \frac{1}{2} \left(\alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2) + (2\alpha_1 + \alpha_2) \right) \tag{3.53}$$

Using the definition (3.53) of the Weyl vector and the above relations for the positive roots we can compute the quadratic Casimir operator $\mathcal{C}_2$ (3.54) for the fundamental representation Λ [140].

$$(\mathcal{C}_2)^k{}_l := \frac{1}{2}(\Lambda, \Lambda + 2\varrho)\, \delta^k{}_l = \frac{1}{2} \cdot \frac{5}{2}\, \delta^k{}_l \tag{3.54}$$

The Casimir operator (3.54) is proportional to the identity because the fundamental representation is irreducible [140]. The expression (3.54) is needed to define a normalisation for (3.52) and the Casimir operator can be related to κ^{-1} by (3.55) [42, 43].

$$(\mathcal{C}_2)^m{}_p = \frac{1}{2}\, (\kappa^{-1})^k{}_l{}^r{}_s\, (T^l{}_k)^m{}_t\, (T^s{}_r)^t{}_p = \frac{N}{2}\, (2n+1)\, \delta^m{}_p \tag{3.55}$$

Comparing (3.54) to (3.55) for the case of sp(4), i.e. $n = 2$, we then find that the normalisation of (3.52) should be $N = \frac{1}{2}$. In analogy to reference [43] we can define the constant $k := 2/(\alpha_1, \alpha_1) = 2$ and σ is the operator that exchanges the factors of the tensor product, which in this case is simply $\delta^k{}_s\, \delta^r{}_l$. Having found the invariant symbol (i.e. the symplectic form Ω), all generators in the fundamental representation, how they act on the representation and the inverse of the Cartan-Killing form, with the correct normalisation, we can now compute the Y-tensor. In general the Y-tensor can be written as (3.56) [43], here we are suppressing the representation indices.

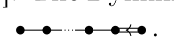
$$\sigma\, Y = k \cdot \left[-(\kappa^{-1})^l{}_m{}^r{}_s\, T^m{}_l\, T^s{}_r + (\Lambda, \Lambda) \right] + \sigma - 1 \tag{3.56}$$

We find that $k \cdot (\Lambda, \Lambda) - 1 = 0$ and the σ term precisely cancels the only other $\delta^k{}_s\, \delta^r{}_l$ term. The Y-tensor of generalised symplectic geometry is hence given by (3.57).

$$Y^{kl}{}_{rs} = \Omega^{kl}\, \Omega_{rs} \tag{3.57}$$

The section condition that follows from this Y-tensor is (3.58).

$$\Omega^{mn}\, \partial_m \otimes \partial_n = 0. \tag{3.58}$$

It has been argued that the section condition in an extended generalised geometry only allows for at most as many coordinates as there are nodes in the longest simply laced chain in the Dynkin diagram of the algebra (that contains the node of the coordinate representation) plus one additional coordinate (because of the regular embedding of GL(n) into the symmetry group GL(n) $\hookrightarrow G$) [34, 43]. The Dynkin diagram associated to to the symplectic algebra sp($2n$) is that of C$_n$: . The longest simply laced chain for C$_n$ is therefore of length $n-1$ which tells us that we can keep at most n, i.e. half of the $2n$ coordinates, which is the same result as in the case of O(n, n). Another way of thinking about equation (3.58) is to look at the decomposition of a general tensor product of two fundamental (coordinate) representations (3.59).

$$\begin{aligned}\partial_m \otimes \partial_n = & \left[\partial_{[m} \otimes \partial_{n]} + \frac{1}{2n}\,\Omega_{mn}\,\Omega^{kl}\partial_k \otimes \partial_l\right] \\ & - \frac{1}{2n}\,\Omega_{mn}\,\Omega^{kl}\partial_k \otimes \partial_l \\ & + \partial_{(m} \otimes \partial_{n)}\end{aligned} \tag{3.59}$$

Each line in (3.59) corresponds to a different irreducible representation of sp$(2n)$ (i.e. the antisymmetric traceless, the trace and the symmetric representations) in the tensor product of two fundamental representations. The section condition (3.58) then removes the one-dimensional trace representation. The existence of the section condition (3.58) illustrates that the extended generalised geometry based on the symplectic algebra sp$(2n)$ is not equivalent to ordinary symplectic geometry, which has a vanishing Y-tensor.

As we have seen in the case of o(n, n), the generalised diffeomorphisms generally do not close into the Lie bracket (cf. equation (3.29)) and one needs to identify an appropriate generalisation (such as the C-bracket), although it is not proven that such a bracket exists for all algebras [42]. Upon application of the section condition the generalised diffeomorphisms reduce to the ordinary gl(n) diffeomorphisms because the Y-tensor vanishes and similarly the generalised bracket reduces to the ordinary Lie bracket because it too only differs by a Y-tensor term. The next step would be the construction of such a generalised bracket for the generalised symplectic diffeomorphisms, but we will now turn our attention to the case of the extended generalised exceptional geometry.

3.5 Exceptional geometry and exceptional field theory

Exceptional field theory (ExFT) is a manifestly $\mathrm{E}_{n(n)}(\mathbb{R})$ invariant Kaluza-Klein-like rewriting and extension of eleven-dimensional supergravity based on an extended generalised exceptional geometry, but without actually truncating any degrees of freedom. ExFT is in some aspects very similar to the gauged supergravity discussed in section 2.3.4 (e.g. regarding the tensor hierarchy [78, 82–87]) and to the Kaluza-Klein rewriting of manifestly O(n, n) invariant DFT described in section 3.3.1 (e.g. regarding the extended generalised geometry).

In 2011 a manifestly $\mathrm{E}_{4(4)}$ invariant canonical formulation of the bosonic sector of eleven-dimensional supergravity, based on a generalised geometry, was published in [141], in [142] an attempt towards an $\mathrm{E}_{5(5)}$ invariant reformulation was made and some other notable early works in this direction are [70, 123, 127, 143–146].
The bosonic $\mathrm{E}_{6(6)}$ ExFT was first published in 2013 [3, 25]. Its supersymmetric completion was first published in [26] and the theory was later reviewed in [36, 103]. The $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{8(8)}$ ExFTs and their supersymmetric completions have been published in [147, 148] and [149, 150] respectively. In the sense of the extended notion of the exceptional groups from table 2.1, the $\mathrm{E}_{2(2)}$ [151], $\mathrm{E}_{3(3)}$ [152], $\mathrm{E}_{4(4)}$ [153] and $\mathrm{E}_{5(5)}$ [154] ExFTs have also been constructed. General reviews of these exceptional field theories were published in [103, 155].
Extending the concepts of ExFT to infinite dimensional Kac-Moody algebras, the E_9 ExFT was constructed in [42, 156, 157]. Already in the early 2000s, before ExFT existed, there have been proposals conjecturing E_{10} [158] or E_{11} [159–161] to be a

n	$\mathrm{E}_{n(n)}$	R_1	R_2	R_3	R_4	R_5	R_6
3	SL(3)×SL(2)	(3,2)	$(\overline{3},1)$	(1,2)	(3,1)	$(\overline{3},\overline{2})$	(8,1) ⊕ (1,3)
4	SL(5)	10	$\overline{5}$	5	$\overline{10}$	24	
5	SO(5,5)	16	10	$\overline{16}$	45		
6	$\mathrm{E}_{6(6)}$	27	$\overline{27}$	78			
7	$\mathrm{E}_{7(7)}$	56	1 ⊕ 133				
8	$\mathrm{E}_{8(8)}$	248	1 ⊕ 248 ⊕ 3875				

TABLE 3.1: The split real forms of the exceptional groups $\mathrm{E}_{n(n)}$ and some of their representations that are relevant in the context of exceptional field theory [103, 147, 149].

symmetry of eleven-dimensional supergravity. Recently an E_{11} master ExFT, containing all other ExFTs and thereby also eleven-dimensional supergravity, has been constructed [137, 162].

The starting point for the construction of the extended generalised exceptional geometry of $\mathrm{E}_{n(n)}$ ($n \leq 8$) exceptional field theory is to locally split the space-time manifold of eleven-dimensional supergravity $\mathcal{M}_{11}$ (3.60) into an external non-compact d-dimensional ($d := 11 - n$) Lorentzian manifold $\mathcal{M}_d^{\text{ext.}}$ and an internal n-dimensional Riemannian manifold $\mathcal{M}_n^{\text{int.}}$. The split (3.60) is in the spirit of Kaluza-Klein theory, however we do not assume any specific topology for the internal manifold $\mathcal{M}_n^{\text{int.}}$ and importantly no degrees of freedom are truncated in this construction.

$$\mathcal{M}_{11} = \mathcal{M}_d^{\text{ext.}} \times \mathcal{M}_n^{\text{int.}} \tag{3.60}$$

The global structure of this exceptional geometry is not known [163] and we will always work in local coordinates. We define the coordinates of the external manifold to be x^μ with $\mu = 0, \ldots, d-1$ and the coordinates of the internal geometry to be y^m with $m = 1, \ldots, n$. Next we need to transform the internal manifold into an extended generalised exceptional geometry. To do so we extend the tangent bundle (e.g. as in (3.18)) and simultaneously add additional auxiliary (or dual) coordinates. As in the doubled geometry the extended coordinates need to be in a representation $R_1(\mathrm{E}_{n(n)})$ of the duality group — in doubled geometry the duality group is $\mathrm{O}(n,n)$ and in exceptional geometry it is $\mathrm{E}_{n(n)}$. We therefore need to add as many auxiliary coordinates to the internal coordinates y^m as are needed to turn them into the extended coordinates Y^M, with $M = 1, \ldots, \dim(R_1)$. The coordinate representations of the exceptional groups $R_1(\mathrm{E}_{n(n)})$ are listed in table 3.1. The extended internal coordinates come with associated coordinate derivatives ∂_M. Overall the coordinates of the external-internal extended geometry are then given by the $d + \dim(R_1)$ many coordinates (x^μ, Y^M). Consistency of the extended generalised exceptional geometry requires the, now $\mathrm{E}_{n(n)}$ covariant, section condition (3.22) to hold. The section condition can be understood as a projection of $\partial_M \otimes \partial_N$ onto the representations $R_2(\mathrm{E}_{n(n)})$, which are listed in table 3.1. Following the argument made in section 3.4 we can deduce that, at most, a subset of n of the $\dim(R_1)$ extended coordinates can survive in any solution of the $\mathrm{E}_{n(n)}$ section condition and therefore solving the section condition effectively removes the internal coordinates that were added.

As is apparent from table 3.1 the dimensions of the representations of the exceptional groups $\mathrm{E}_{n(n)}$ are very different depending on the rank n and the invariant symbols that the representations admit are therefore also very different. This is in contrast to the representations and invariant symbols of $\mathrm{O}(n,n)$, for which we could write down formulas that are valid for any rank and therefore we could describe doubled geometry and doubled field theory in a unified form for any n. Due to the diversity and irregularity of the dimensions of the representations of $\mathrm{E}_{n(n)}$ it is not possible to formulate all aspects of exceptional geometry and exceptional field theory in a form that is true for all ranks. We will therefore focus on the $(5+27)$-dimensional $\mathrm{E}_{6(6)}$ exceptional field theory in the remainder of this thesis. We choose to consider the $\mathrm{E}_{6(6)}$ ExFT because it is, in some sense, the simplest case of the true exceptional groups ($n = 6, 7, 8$). Because the external space-time is odd-dimensional there are no self-dual differential forms in $\mathrm{E}_{6(6)}$ ExFT and therefore it has a true Lagrangian — unlike the $\mathrm{E}_{7(7)}$ ExFT, in which the one-forms are (twisted) self-dual and therefore it only has a pseudo-Lagrangian. And unlike in the $\mathrm{E}_{8(8)}$ ExFT there are no constrained compensator fields in the $\mathrm{E}_{6(6)}$ theory. As was stated earlier we furthermore focus on the bosonic sector of the theory for simplicity.

The outline of the remainder of this chapter is as follows. In section 3.5.1 we make a brief excursion to describe the Lie algebra $\mathfrak{e}_6$. In section 3.5.2 we discuss the $\mathrm{E}_{6(6)}$ extended generalised exceptional geometry in detail. In section 3.5.3 we discuss the $\mathrm{E}_{6(6)}$-covariant field theory constructed on this extended geometry, its gauge transformations are discussed in section 3.5.4 and its gauge algebra is discussed in 3.5.5. In section 3.5.6 we discuss how the section condition can be solved and how the section condition relates ExFT to eleven-dimensional supergravity. Applications of ExFT are briefly discussed in section 3.5.7. In the section 3.5.8 we construct the explicit non-integral form of the Lagrangian topological term of the $\mathrm{E}_{6(6)}$ ExFT.

Most of the remainder of this chapter is based on and closely follows the structure of parts of the publication [41], which in particular reviews the results of [3, 25, 36].

3.5.1 Excursion: $\mathfrak{e}_6$ Lie algebra

The Lie algebra $\mathfrak{e}_6$ is a rank 6 exceptional Lie algebra (in the sense of the Cartan classification [68]) of real dimension 78. Its Dynkin diagram is shown in figure 3.1. Using the Iwasawa (Cartan) decomposition the $\mathfrak{e}_6$ algebra can be decomposed as

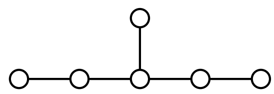


FIGURE 3.1: The Dynkin diagram of the Lie algebra $\mathfrak{e}_6$

$\mathfrak{e}_6 = \mathfrak{usp}_8 \oplus \mathfrak{p}$, into the algebra of USp(8), which is the maximal compact subgroup of $\mathrm{E}_{6(6)}$, and the non-compact complement $\mathfrak{p}$. We can therefore explicitly write the $\mathfrak{e}_6$ algebra in terms of the 36 generators of $\mathfrak{usp}_8$, which we call $X^{AB} = X^{(AB)}$, with $A, B, ... = 1, \ldots, 8$ and the 42 fully antisymmetric generators of the non-compact complement $\mathfrak{p}$, which we call $Y^{ABCD} = Y^{[ABCD]}$. The full $\mathfrak{e}_6$ algebra can then be written in terms of the generators X^{AB} and Y^{ABCD} as the Lie bracket relations (3.61), (3.62) and (3.63). The δ-symbols with several indices here signify (anti-)symmetrised products of the δ^A_B-symbols. Ω is the symplectic form of $\mathfrak{usp}_8$. The first relation (3.61) is simply the $\mathfrak{sp}_8$ subalgebra.

$$[X^{AB}, X^{CD}] = 4\ X^{\tilde{A}\tilde{C}}\Omega^{\tilde{B}\tilde{D}} \quad \cdot (\delta^{(AB)}_{\tilde{A}\tilde{B}}\delta^{(CD)}_{\tilde{C}\tilde{D}}) \tag{3.61}$$

$$[X^{AB}, Y^{CDEF}] = -8\ \Omega^{\tilde{C}\tilde{B}}Y^{\tilde{A}\tilde{D}\tilde{E}\tilde{F}} \quad \cdot (\delta^{(AB)}_{\tilde{A}\tilde{B}}\delta^{[CDEF]}_{\tilde{C}\tilde{D}\tilde{E}\tilde{F}}) \tag{3.62}$$

$$\begin{aligned}[Y^{ABCD}, Y^{EFGH}] = 24^2\ &(X^{\tilde{A}\tilde{E}}\Omega^{\tilde{B}\tilde{F}}\Omega^{\tilde{C}\tilde{G}}\Omega^{\tilde{D}\tilde{H}} - \frac{3}{4}X^{\tilde{A}\tilde{E}}\Omega^{\tilde{B}\tilde{C}}\Omega^{\tilde{F}\tilde{G}}\Omega^{\tilde{D}\tilde{H}}) \\ &\cdot (\delta^{[ABCD]}_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}\delta^{[EFGH]}_{\tilde{E}\tilde{F}\tilde{G}\tilde{H}})\end{aligned} \tag{3.63}$$

We will not need to make use of the explicit form of the $\mathfrak{e}_6$ algebra in the following, but it may help conceptually to keep this result in mind.

3.5.2 $\mathrm{E}_{6(6)}$ extended generalised exceptional geometry

We can think of the (internal) extended 27-dimensional $\mathrm{E}_{6(6)}$ generalised exceptional geometry as the coordinate-extended version of the generalised exceptional geometry (cf. section 3.2.1) associated to the vector bundle (3.64), with $\mathrm{T}(\mathcal{M}_6^{\text{int.}})$ being the standard tangent bundle of the (unextended) internal Riemannian manifold. The (internal) extended coordinates Y^M, are in the fundamental representation $R_1(E_{6(6)}) = 27$, with index $M = 1, \dots, 27$.

$$\mathrm{E} := \mathrm{T}(\mathcal{M}_6^{\text{int.}}) \oplus \Lambda^2\mathrm{T}^*(\mathcal{M}_6^{\text{int.}}) \oplus \Lambda^5\mathrm{T}^*(\mathcal{M}_6^{\text{int.}}) \tag{3.64}$$

In this section we review some of the results of [25]. Before we can begin to explicitly describe the extended $\mathrm{E}_{6(6)}$ exceptional geometry we first need to introduce the $\mathrm{E}_{6(6)}$-invariant d-symbols. The invariant symbols d_{LMN} and d^{LMN} carry three, fully symmetric, indices in the (anti-)fundamental 27 (or $\overline{27}$) representation of $\mathrm{E}_{6(6)}$ [25, 74]. Up to the normalisation, they are the unique invariant symbols of the fundamental representation of $\mathrm{E}_{6(6)}$. The invariance condition can be written as (3.65). We choose the normalisation of the symbols to be determined by the requirement for them to be inverse to each other (3.66). Furthermore the d-symbols obey the cubic identities (3.67) and (3.68), which are required in some calculations, because they allow us to move $\mathrm{E}_{6(6)}$ indices between objects.

$$d^{KLM}\, M_{KN}\, M_{LR}\, M_{MS} = d_{NRS} \tag{3.65}$$

$$d^{MKL}\, d_{KLN} = \delta^M_N \tag{3.66}$$

$$d_{S(MN}\, d_{PQ)T}\, d^{STR} = \frac{2}{15}\,\delta^R_{(M}\, d_{NPQ)} \tag{3.67}$$

$$d_{STR}\, d^{S(MN}\, d^{PQ)T} = \frac{2}{15}\,\delta^{(M}_R\, d^{NPQ)} \tag{3.68}$$

We label the adjoint representation by $\zeta = 1, \dots, 78$ and take t_ζ to be the generators of the algebra $\mathfrak{e}_6$. We can then express the generators in the fundamental representation as $(t_\zeta)^M{}_N$. The Cartan-Killing form is proportional to the trace $(t_{\zeta_1})^N{}_M\,(t_{\zeta_2})^M{}_N$ and we can use this to raise and lower the adjoint indices. The projector $\mathbb{P}^M{}_N{}^K{}_L$ onto the adjoint representation is defined by (3.69) and its normalisation is given by $\mathbb{P}^M{}_N{}^N{}_M = 78$. The adjoint projector can be expressed explicitly in terms of the invariant symbols as in (3.70).

$$\mathbb{P}^M{}_N{}^K{}_L := (t_\zeta)^M{}_N\,(t^\zeta)^K{}_L \tag{3.69}$$

$$\mathbb{P}^M{}_N{}^K{}_L = \frac{1}{18}\,\delta^M_N\,\delta^K_L + \frac{1}{6}\,\delta^M_L\,\delta^K_N - \frac{5}{3}\,d^{MKR}\,d_{RNL} \tag{3.70}$$

The $E_{6(6)}$ Y-tensor, from which we can derive the structure of the $E_{6(6)}$ exceptional geometry, is given by (3.71). We can also arrive at this Y-tensor by using the relation (3.26), with the $E_{6(6)}$ special weight $\omega = -1/3$ (for a general $E_{n(n)}$ it is $\omega = \frac{1}{n-9}$) and the coefficient $\alpha = 6$ and inserting the explicit expression of the adjoint projector (3.70) [103, 127].

$$Y^{MK}{}_{NL} = 10\, d^{MKR}\, d_{RNL} \tag{3.71}$$

The $E_{6(6)}$ covariant section condition, which follows from (3.22) with the Y-tensor (3.71), is given by (3.72).

$$d^{KLM}\, \partial_L \otimes \partial_M = 0 \tag{3.72}$$

The section condition (3.72) is interpreted as the two conditions (3.73) where Ψ, Φ are arbitrary functions, which includes in particular all fields and gauge parameters. In this sense the section condition is not a constraint on any particular function, but we should think of it as a condition on the extended coordinates themselves.

$$d^{KLM}\, \partial_L \Phi\, \partial_M \Psi = 0, \quad d^{KLM}\, \partial_L \partial_M \Phi = 0 \tag{3.73}$$

Unlike the section condition of double field theory, which can be seen as (partially) originating from the level-matching condition of string theory, the section condition of ExFT is (physically) postulated ad hoc by analogy. In some situations it may be possible to interpret the section condition of ExFT as coming from 1/2-BPS constraints [33, 164]. The section condition also appears as a consistency condition for the closure of the generalised diffeomorphism algebra, just like in doubled geometry. The $E_{6(6)}$ section condition furthermore implies that at most 6 of the 27 coordinates exist in any solution of the section condition, which is equal to the number of the original internal (physical) coordinates of eleven-dimensional supergravity.

The $E_{6(6)}$ generalised Lie derivative $\mathbb{L}_\Lambda$, with parameter Λ^M, of a generalised vector V^M, can be written in terms of the adjoint projector (3.70) as (3.74). The constant $\lambda(V) \in \mathbb{R}$ is the generalised weight of the generalised vector V^M. This expression is in form analogous to the generalised Lie derivative in doubled geometry (3.23), but here the second term is interpreted as an $E_{6(6)}$ transformation.

$$\mathbb{L}_\Lambda V^M := \Lambda^K\, \partial_K V^M - 6\, \mathbb{P}^M{}_N{}^K{}_L\, \partial_K \Lambda^L\, V^N + \lambda(V)\, \partial_N \Lambda^N\, V^M \tag{3.74}$$

The generalised Lie derivative of a generalised covector W_M, of weight $\lambda(W)$, can equivalently be written as (3.75). Because the generalised Lie derivative satisfies the Leibniz rule these expressions extend in the standard fashion to arbitrary generalised tensors.

$$\mathbb{L}_\Lambda W_M := \Lambda^K\, \partial_K W_M + 6\, \mathbb{P}^N{}_M{}^K{}_L\, \partial_K \Lambda^L\, W_N + \lambda(W)\, \partial_N \Lambda^N\, W_M \tag{3.75}$$

Parameters of the generalised Lie derivative that are of the form (3.76), with W_K being an arbitrary generalised covector, lead to a vanishing generalised Lie derivative on any other function, when the section condition is applied. Therefore we refer to parameters of the form (3.76) as trivial.

$$\Lambda^M = d^{MNK}\, \partial_N W_K \tag{3.76}$$

We can furthermore verify the compatibility of the above generalised Lie derivative with the $E_{6(6)}$ invariant d-symbols and find that they transform as constants (3.77).

$$\mathbb{L}_\Lambda d_{MNK} = 0 \tag{3.77}$$

In the extended generalised exceptional geometry the Lie bracket needs to be modified by an additional Y-tensor contribution (in analogy to the C-bracket of DFT), according to (3.31), and we call the modified bracket the E-bracket. Inserting the explicit expression for the $E_{6(6)}$ Y-tensor (3.71) into the general formula (3.31) we find that we can write the E-bracket of two generalised vectors Λ_1^M, Λ_2^N explicitly as (3.78).

$$[\Lambda_1, \Lambda_2]_{\mathrm{E}}^M := 2\,\Lambda_{[1}^K\,\partial_K\Lambda_{2]}^M - 10\,d^{MNP}\,d_{KLP}\,\Lambda_{[1}^K\,\partial_N\Lambda_{2]}^L \tag{3.78}$$

It can be verified, in a somewhat lengthy calculation, that the commutator of two generalised Lie derivatives can again be written as a generalised Lie derivative, i.e. the generalised Lie derivative obeys the algebra (3.79). The parameter of the resulting generalised Lie derivative is given by the E-bracket of the original parameters. This relation holds true acting on any function, however only up to terms that vanish when the section condition (3.72) is applied. We can thus think of the section condition as a consistency condition, in the sense that it is required by the closure of the generalised diffeomorphism algebra and it arises naturally in this context. In the calculation of the algebra (3.79) the cubic identities (3.67) and (3.68) have to be applied repeatedly, in order to move $E_{6(6)}$ indices between objects.

$$[\mathbb{L}_{\Lambda_1}, \mathbb{L}_{\Lambda_2}] = \mathbb{L}_{[\Lambda_1,\Lambda_2]_{\mathrm{E}}} \tag{3.79}$$

The E-bracket is bilinear, antisymmetric and obeys the Leibniz rule, hence it defines a Leibniz (or Loday) algebra [165–167]. But the E-bracket is not itself a Lie bracket because the Jacobi identity holds only up to a trivial parameter term, i.e. the Jacobiator (3.80) is of trivial form, where U^M, V^N, W^K are generalised vectors.

$$J(U, V, W) := [[U, V]_{\mathrm{E}}\,, W]_{\mathrm{E}} + [[W, U]_{\mathrm{E}}\,, V]_{\mathrm{E}} + [[V, W]_{\mathrm{E}}\,, U]_{\mathrm{E}} \tag{3.80}$$

We can define the bracket (3.81), which is in analogy to the Dorfman bracket of generalised (complex) geometry or the D-bracket of doubled geometry [25].

$$(V \circ W)^M := \mathbb{L}_V W^M \tag{3.81}$$

Sometimes it is beneficial to trade the antisymmetry of the E-bracket for the Jacobi identity and indeed the Dorfman bracket (3.81) satisfies the Jacobi identity, but it is not antisymmetric, instead the relation (3.82) holds, if $\lambda(W) = 1/3$.

$$(V \circ W)^M = [V, W]_{\mathrm{E}}^M + 5\,d^{MKR}\,\partial_K(d_{RPL}\,V^P\,W^L) \tag{3.82}$$

The antisymmetric part of (3.82) is identical to the E-bracket, but there is an additional symmetric term, which is of trivial form. In some calculations the symmetrised Dorfman bracket is a useful expression and we will make use of it in chapter 6.

Written in the form of (3.82) the Dorfman bracket can be seen as an analogy to the split (2.29) of the embedding structure constants $X_{MN}{}^K$ of gauged supergravity. Just like the antisymmetric part of $X_{MN}{}^K$ the E-bracket does not satisfy the Jacobi identity. As we will see in section 3.5.3, when discussing the exceptional field theory, this leads to further similarities between these theories. At least in part the section condition in ExFT can be seen as being analogous to the quadratic constraints of the embedding tensor in gauged supergravity [57].

3.5.3 The Lagrangian of $E_{6(6)}$ exceptional field theory

Having discussed the extended $E_{6(6)}$ generalised exceptional geometry we can now turn to the exceptional field theory that is build upon it. In this section we summarise some of the main findings of the references [3, 25] concerning the Lagrangian formulation of the bosonic sector of $E_{6(6)}$ ExFT. We begin with the description of the bosonic field content of the $E_{6(6)}$ ExFT, which is given by (3.83).

$$\{E_\mu{}^\alpha, M_{MN}, A_\mu^M, B_{\mu\nu M}\} \tag{3.83}$$

All of the fields (3.83) and all of the gauge parameters of the $E_{6(6)}$ ExFT depend on all of the 5+27 external and internal coordinates (x^μ, Y^M). The $\alpha, \beta = 0, \ldots, 4$ are external five-dimensional flat Lorentz indices, $\mu, \nu = t, 1, \ldots, 4$ are external five-dimensional curved space-time indices and $M, N = 1, \ldots, 27$ are the (anti-)fundamental $E_{6(6)}$ indices. The external vielbein $E_\mu{}^\alpha$ is related to the external metric $G_{\mu\nu}$ by (3.84), with $\eta_{\alpha\beta}$ being the external Minkowski metric with signature $(-++++)$.

$$G_{\mu\nu} = E_\mu{}^\alpha\, E_\nu{}^\beta\, \eta_{\alpha\beta} \tag{3.84}$$

The fields $M_{MN} = M_{(MN)}$ can be interpreted either, from the internal perspective, as generalised $E_{6(6)}$ metric components or from the external perspective, as scalar fields. The inverse internal metric is defined by $M_{MK}\, M^{KN} = \delta_M^N$. The M_{MN} parametrise the $E_{6(6)}/\mathrm{USp}(8)$ coset, which is 42-dimensional (cf. table 2.1). This moreover implies that only 42 of its 378 symmetric components are independent. We refer to the relations among the 378 components, implied by the coset structure, as the coset constraints of the scalar fields. Among other things the coset constraints imply that $\det(M_{MN}) = 1$. The one- and two-form fields A_μ^M and $B_{\mu\nu M}$ each carry one additional (anti-)fundamental $E_{6(6)}$ index — thus they are (co-)vectors from the internal perspective.

In order to understand why the bosonic field content of the $E_{6(6)}$ ExFT is given by (3.83) and how these fields relate to the degrees of freedom of eleven-dimensional supergravity we need to consider the Kaluza-Klein-like $(5+6)$-dimensional split (3.60) of eleven-dimensional supergravity. We decompose the eleven-dimensional space-time indices as $\hat{\mu} = (\mu, m)$, with $\mu = t, 1, \ldots, 4$ and $m = 1, \ldots, 6$. In section 2.2.1 we have seen that the bosonic field content of eleven-dimensional supergravity is a metric $G_{\hat{\mu}\hat{\nu}}$ and a three-form $C_{\hat{\mu}\hat{\nu}\hat{\rho}}$. Now we need to decompose the metric and the three-form according to the $(5+6)$-dimensional split into their components $G_{\mu\nu}, G_{\mu n}, G_{mn}$ and $C_{\mu\nu\rho}, C_{\mu\nu r}, C_{\mu nr}, C_{mnr}$. These components can now be rearranged. As was explained in section 2.3.3, we can Hodge-dualise the field strength of a p-form in d dimensions to yield an equivalent $(d-p-2)$-form. One has to dualise all forms, with respect to the external geometry, to the lowest possible degree to arrive at the field content (3.83). The fully external part of the metric $G_{\mu\nu}$ becomes the external metric of ExFT. The independent components of the scalar fields M_{MN} comprise the $42 = 21 + 1 + 20$ scalar fields coming from G_{mn}, $C_{\mu\nu\rho}$ and C_{mnr}. The vector fields A_μ^M consist of the $27 = 6+6+15$ vector fields coming from $G_{\mu n}$, $C_{\mu\nu r}$ and $C_{\mu nr}$. There are now no more degrees of freedom left to make the two-forms $B_{\mu\nu M}$, but the two-forms are needed in $E_{6(6)}$ ExFT to make the one-forms transform covariantly. In five dimensions the 27 one-forms A_μ^M are dual to the 27 two-forms $B_{\mu\nu M}$ and we can hence introduce them as (on-shell) dual topological fields. Because the two-forms are dual to the one-forms no new propagating degrees of freedom are introduced. The duality relation is given dynamically by the two-form equations of motion. Overall we have thus found one metric, 42 scalars, 27 one-forms and 27 two-forms, which agrees with the field content (3.83).

In ExFT the one-forms A_μ^M act as the gauge connection for the generalised diffeomorphism symmetry. Because the generalised diffeomorphism parameters $\Lambda^M(x,Y)$ also depend on the external coordinates we need to introduce the external covariant derivative defined by (3.85).

$$\mathcal{D}_\mu := \partial_\mu - \mathbb{L}_{A_\mu} \tag{3.85}$$

We define the one-forms to transform, under a generalised diffeomorphism with parameter Λ^M, as the covariant derivative of the gauge parameter (3.86). We can furthermore think of (3.86) as the covariantised version of an abelian $\mathrm{U}(1)^{27}$ gauge transformation $\delta_\Lambda A_\mu^M = \partial_\mu \Lambda^M$.

$$\delta_\Lambda A_\mu^M := \mathcal{D}_\mu \Lambda^M \tag{3.86}$$

We can define the naive field strength (3.87), which replaces the Lie bracket with the E-bracket (3.78). This field strength is analogous to the field strength (2.30) in gauged maximal supergravity.

$$F_{\mu\nu}^M := 2\,\partial_{[\mu} A_{\nu]}^M - [A_\mu, A_\nu]_{\mathrm{E}}^M \tag{3.87}$$

In analogy to gauged maximal supergravity, the naive field strength (3.87) fails to transform covariantly. Instead it transforms as (3.88) (cf. equation (2.32)) with an additional trivial non-covariant term, which is due to the non-vanishing Jacobiator of the E-bracket.

$$\delta F_{\mu\nu}^M = 2\,\mathcal{D}_{[\mu}\,\delta A_{\nu]}^M + 10\,d^{MKR}\,d_{NLR}\,\partial_K\left(A_{[\mu}^N\,\delta A_{\nu]}^L\right) \tag{3.88}$$

The relation (3.88) furthermore implies (3.89).[11]

$$\delta_\Lambda F_{\mu\nu}^M = \mathbb{L}_\Lambda F_{\mu\nu}^M + 10\,d^{MKR}\,d_{NLR}\,\partial_K\left(-\Lambda^N\,F_{\mu\nu}^L + A_{[\mu}^N\,\delta A_{\nu]}^L\right) \tag{3.89}$$

Continuing with the analogy to gauged supergravity we can solve this issue by introducing the two-forms $B_{\mu\nu M}$, which we can define to transform in such a way that they absorb the offending non-covariant term. There are not naturally any two-forms present in the field content coming from eleven-dimensional supergravity, but we can Hodge-dualise the one-forms (via their field strength) to yield the two-forms $B_{\mu\nu M}$ that we need, as was discussed above. In analogy to the covariant field strength (2.33) of gauged supergravity we then add a Stückelberg-type coupling term to the naive one-form field strength (3.87) to arrive at the covariant field strength (3.90).

$$\begin{aligned}\mathcal{F}_{\mu\nu}^M &:= F_{\mu\nu}^M + 10\,d^{KLM}\,\partial_K B_{\mu\nu L} &(3.90)\\ &= 2\,\partial_{[\mu} A_{\nu]}^M - [A_\mu, A_\nu]_{\mathrm{E}}^M + 10\,d^{KLM}\,\partial_K B_{\mu\nu L} &(3.91)\end{aligned}$$

Comparing the improved covariant field strengths of gauged supergravity (2.33) and ExFT (3.90), we can see that the E-bracket takes the place of the antisymmetric part of the embedding structure constants and the d-symbol contracted with an internal

[11] To show (3.89) one can make use of the relations (3.86), (3.94) and the symmetrised Dorfman product (3.82).

derivative $d^{KLM}\,\partial_K$ take the place of the symmetric part of the embedding structure constants.

The improved field strength (3.90) does indeed transform covariantly as (3.92).

$$\delta\mathcal{F}^M_{\mu\nu} = 2\,\mathcal{D}_{[\mu}\,\delta A^M_{\nu]} + 10\,d^{MNK}\,\partial_K \Delta B_{\mu\nu N} \tag{3.92}$$

In (3.92) we have used the modified transformation $\Delta B_{\mu\nu N}$ of the two-forms, which is defined as (3.93) in order to cancel the non-covariant term in (3.88).

$$\Delta B_{\mu\nu N} := \delta B_{\mu\nu N} + d_{NKL}\,A^K_{[\mu}\,\delta A^L_{\nu]} \tag{3.93}$$

The commutator of the external covariant derivatives (3.94) generates the one-form field strength. Because the covariant derivative is of the form (3.85) the field strength appears as the parameter of a generalised Lie derivative. The Stückelberg coupling term in the covariant field strength (3.90) is of trivial form and hence the commutator (3.94) cannot distinguish between the naive and the covariant field strengths.

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathbb{L}_{F_{\mu\nu}} = -\mathbb{L}_{\mathcal{F}_{\mu\nu}} \tag{3.94}$$

The field strength $\mathcal{H}_{\rho\sigma\tau N}$ of the two-forms $B_{\mu\nu M}$ can be written as (3.95), where the "..." indicate terms that vanish under the projection $d^{MNK}\,\partial_K$.

$$\mathcal{H}_{\rho\sigma\tau N} := 3\,\mathcal{D}_{[\rho}B_{\sigma\tau]N} - 3\,d_{NKL}A^K_{[\rho}\left(\partial_\sigma A^L_{\tau]} - \frac{1}{3}[A_\sigma, A_{\tau]}]^L_E\right) + \dots \tag{3.95}$$

The two-form field strength (3.95) can be found by solving the Bianchi identity (3.96).

$$3\,\mathcal{D}_{[\mu}\mathcal{F}^M_{\nu\rho]} = 10\,d^{MNK}\,\partial_K\mathcal{H}_{\mu\nu\rho N} \tag{3.96}$$

The (topological) two-forms were introduced in order for the one-form field strength (3.90) to transform covariantly. This was in analogy to the tensor hierarchy of gauged supergravity. Similarly the two-form field strength (3.95) does not transform covariantly without the introduction of a three-form and so on [25, 103]. Luckily the three-form terms in (3.95), which are contained in the "...", are projected out in the $E_{6(6)}$ ExFT and do not appear in the Lagrangian.

The bosonic $E_{6(6)}$ exceptional field theory action is given by (3.97).

$$S_{\text{ExFT}} = \int d^5x \int d^{27}Y\,\mathcal{L}_{\text{ExFT}} \tag{3.97}$$

As in double field theory, the action (3.97) should be thought of as a symmetric and effective way of encoding the classical equations of motion. It is not known how the integral over the internal geometry can be carried out explicitly, in a meaningful way, before the section condition (3.72) is solved. We should therefore regard the internal integral in the ExFT action as being symbolic.
The bosonic $E_{6(6)}$ ExFT Lagrangian $\mathcal{L}_{\text{ExFT}}$ consists of the five distinct terms (3.98).

$$\mathcal{L}_{\text{ExFT}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{sc}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{top}} \tag{3.98}$$

The first term in the Lagrangian is the improved Einstein-Hilbert term $\mathcal{L}_{\text{EH}}$ (3.99). $\mathcal{L}_{\text{EH}}$ consists of the standard Einstein-Hilbert term $E\,R$, where $E := \det({E_\mu}^\alpha)$ is the vielbein determinant and R is the $\mathcal{D}_\mu$-covariantised Ricci scalar of ${E_\mu}^\alpha$. What is meant by $\mathcal{D}_\mu$-covariantised is that all (external) partial derivatives in the Ricci scalar are replaced by covariant derivatives $\mathcal{D}_\mu$. Because of the covariantisation of the Ricci scalar and because of the non-commutativity of the covariant derivatives (3.94) one needs an additional one-form-dependent improvement term in order for the covariantised Riemann tensor to transform tensorially under local Lorentz transformations. $\mathcal{L}_{\mathrm{EH}}$ is analogous to the covariantised Einstein-Hilbert term in the Kaluza-Klein-like reformulation of DFT (3.42).

$$\mathcal{L}_{\mathrm{EH}} = E\hat{R} = ER + E\,\mathcal{F}^M_{\alpha\beta}\,E^{\alpha\rho}\,\partial_M E^\beta_\rho \tag{3.99}$$

$$\mathcal{L}_{\mathrm{sc}} = \frac{E}{24}\,G^{\mu\nu}\,\mathcal{D}_\mu M_{MN}\,\mathcal{D}_\nu M^{MN} \tag{3.100}$$

$$\mathcal{L}_{\mathrm{pot}} = -E\,V_{\mathrm{pot}}(G_{\mu\nu}, M_{MN}) \tag{3.101}$$

$$\mathcal{L}_{\mathrm{YM}} = -\frac{E}{4}\,M_{MN}\mathcal{F}^M_{\mu\nu}\,\mathcal{F}^{\mu\nu N} \tag{3.102}$$

The second term in (3.98) is the scalar kinetic term $\mathcal{L}_{\mathrm{sc}}$ (3.100). Moreover we can think of (3.100) as a covariantised $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ non-linear coset sigma model. The third term in the Lagrangian is the scalar potential term $\mathcal{L}_{\mathrm{pot}}$ (3.101). The potential $V_{\mathrm{pot}}(G_{\mu\nu}, M_{MN})$ itself is explicitly given by (3.103) and only depends on the external metric ($G := \det(G_{\mu\nu})$), the scalar fields and internal coordinate derivatives. It does not depend on the external derivatives and hence the name potential is justified.

$$\begin{aligned} V_{\mathrm{pot}} = &-\frac{1}{24}\,M^{MN}\,\partial_M M^{KL}\,\partial_N M_{KL} + \frac{1}{2}\,M^{MN}\,\partial_M M^{KL}\,\partial_L M_{NK} \\ &- \frac{1}{2}\,G^{-1}\,\partial_M G\,\partial_N M^{MN} - \frac{1}{4}\,M^{MN}\,G^{-1}\,\partial_M G\,G^{-1}\,\partial_N G \\ &- \frac{1}{4}\,M^{MN}\,\partial_M G^{\mu\nu}\,\partial_N G_{\mu\nu} \end{aligned} \tag{3.103}$$

The fourth term in (3.98) is the (generalised) Yang-Mills term $\mathcal{L}_{\mathrm{YM}}$ (3.102), which is of the standard Yang-Mills form, but using the covariant field strength (3.90). The $\mathrm{E}_{6(6)}$ indices are contracted by the internal generalised metric M_{MN}.
The fifth and final term in the Lagrangian is the topological term $\mathcal{L}_{\mathrm{top}}$, whose action can be written as the $(6+27)$-dimensional integral over an exact six-form (3.105) [25]. Here we have used the definitions $\mathcal{F}^M := \frac{1}{2}\mathcal{F}^M_{\mu\nu}\,dx^\mu \wedge dx^\nu$ and $\mathcal{H}_M := \frac{1}{3!}\mathcal{H}_{\mu\nu\rho M}\,dx^\mu \wedge dx^\nu \wedge dx^\rho$. In the Lagrangian formulation the explicit non-integral form $\mathcal{L}_{\mathrm{top}}$ of the topological term in $5+27$ dimensions, which is not manifestly gauge-invariant, is not needed. We construct the explicit non-integral topological term $\mathcal{L}_{\mathrm{top}}$ in section 3.5.8, because the explicit form of this term is required in the Hamiltonian formalism in order to carry out the Legendre transformation of the Lagrangian (3.98).

$$\mathcal{S}_{\mathrm{top}} = \int d^{27}Y \int d^5x\,\mathcal{L}_{\mathrm{top}} \tag{3.104}$$

$$= \kappa \int d^{27}Y \int_{\mathcal{M}_6} \big(d_{MNK}\,\mathcal{F}^M \wedge \mathcal{F}^N \wedge \mathcal{F}^K - 40\,d^{MNK}\mathcal{H}_M \wedge \partial_N \mathcal{H}_K\big) \tag{3.105}$$

We should note that the only kinetic term of the two-forms, i.e. the only appearance of the two-form field strength, is inside the topological term and thus the two-forms are indeed topological degrees of freedom. In contrast the field strength of the one-forms appears in the Yang-Mills term, in the topological term and in the Einstein-Hilbert

Weight λ	Objects
$-2/3$	$G^{\mu\nu}$, $\hat{R}$, V_{pot}
$-1/3$	∂_M, $E_\alpha{}^\mu$
0	∂_μ, $\mathbb{L}_{A_\mu}$, d_{MNK}, M_{MN}, $\hat{R}_{\mu\nu}{}^{\alpha\beta}$
$1/3$	A_μ^M, $\mathcal{F}_{\mu\nu}^M$, Λ^M, $E_\mu{}^\alpha$
$2/3$	$B_{\mu\nu M}$, $\Xi_{\mu M}$, $\mathcal{H}_{\mu\nu\rho M}$, $G_{\mu\nu}$
1	$\mathcal{L}_{\text{ExFT}}$
$5/3$	E

TABLE 3.2: The exceptional generalised diffeomorphism weights of the most important objects of $E_{6(6)}$ exceptional field theory.

improvement term. The two-forms couple to all of these terms too due to the Stückelberg coupling term in the one-form field strength (3.90).

In the topological action (3.105) the overall constant of the topological term κ is introduced. In this thesis we will keep κ general and not insert its value, because this allows us to track where the terms of the topological Lagrangian are going. If we were to insert its numerical value it would be given by $\kappa = +\sqrt{10}/6$ [25, 26, 36, 155].[12] Only the modulus of κ, but not the sign, is fixed by the symmetries in the bosonic ExFT and hence the sign is conventional [26].

3.5.4 Lagrangian gauge transformations

We can now discuss the (infinitesimal) gauge transformations that leave the action (3.97) invariant. This section reviews some of the results of [25].

Each of the terms of the Lagrangian (3.98) is individually invariant under generalised diffeomorphisms. From the perspective of eleven-dimensional supergravity the generalised diffeomorphisms combine the spatial diffeomorphisms of the six original internal dimensions with three-form gauge transformations. In this sense the three-form gauge transformations have become geometrised.
With the exception of the p-forms the fields of ExFT transform under generalised diffeomorphisms, with parameter Λ, as the generalised Lie derivative $\delta_\Lambda = \mathbb{L}_\Lambda$, including the appropriate weight terms, with the generalised diffeomorphism weights listed in table 3.2. The transformations of the p-forms are modified due to the tensor hierarchy and the Stückelberg coupling. By definition the transformation of the one-forms (3.106) combines the abelian $U(1)^{27}$ gauge transformations with the generalised Lie derivative to transform as the covariant derivative of the gauge parameter Λ^M. The Stückelberg coupling term in the one-form field strength (3.90) moreover induces a

[12] In the ExFT literature there is some confusion regarding the coefficient of the topological term which stems from a hidden rescaling of κ in reference [25] — if we are aware of this fact and do not mix up the different values of the constants then the calculations of [25] are nonetheless all correct. We define $\kappa_1 := \sqrt{10}/6$ and $\kappa_2 := \frac{3}{4}\kappa_1 = \sqrt{10}/8$ to distinguish the rescaling. The value of κ_1 is used in the equations (3.7), (3.8) and (3.9) of [25]. But starting with equation (3.29) of [25], where the numerical value of this coefficient is determined and in particular in the equation (3.31), which states $\kappa^2 = \frac{5}{32}$, the rescaled coefficient κ_2 is used. As can be seen from equation (3.8) of [25] this additional factor of $\frac{3}{4}$ comes from taking the variation of the topological term and has been absorbed into κ_2. The references [26, 155] continue to use the rescaled κ_2 consistently, while the reference [36] uses the original κ_1 consistently.

two-form tensor gauge transformation, with parameter $\Xi_{\mu N}$, in the one-forms (3.106).

$$\delta A_\mu^M = \mathcal{D}_\mu \Lambda^M - 10\, d^{MNK}\, \partial_K \Xi_{\mu N} \tag{3.106}$$

$$\Delta B_{\mu\nu M} = 2\, \mathcal{D}_{[\mu} \Xi_{\nu]M} + d_{MKL}\, \Lambda^K\, \mathcal{F}_{\mu\nu}^L + \mathcal{O}_{\mu\nu M} \tag{3.107}$$

With the modified transformation $\Delta B_{\mu\nu M}$, defined in (3.93), the two-forms transform under tensor gauge transformations, generalised diffeomorphisms and shifts as (3.107). The two-form tensor gauge transformation, with parameter $\Xi_{\nu M}$, is of the standard form, but covariantised. Due to (3.93) and (3.89) the two-forms transform under generalised diffeomorphisms, in this parametrisation, not as the generalised Lie derivative but as in (3.107). In addition there is a shift transformation with parameter $\mathcal{O}_{\mu\nu M}$ and the parameter vanishes under the projection (3.108), which is consistent with the Stückelberg coupling and the properties of the two-form field strength.

$$d^{MNK}\, \partial_K \mathcal{O}_{\mu\nu M} = 0 \tag{3.108}$$

The ExFT action (3.97) is furthermore invariant under external diffeomorphisms. The parameter $\xi^\mu(x, Y)$ of the external diffeomorphisms depends on all of the coordinates — like every other gauge parameter — but the external diffeomorphism symmetry is only manifest for parameters that do not depend on the internal coordinates, i.e. $\partial_M \xi^\mu = 0\ \forall M$. If the gauge parameter depends non-trivially on the internal coordinates, i.e. $\exists M : \partial_M \xi^\mu \neq 0$, then the external diffeomorphism transformations connect all terms in the Lagrangian (3.98) and requiring invariance of the action fixes all relative coefficients in (3.98).

The bosonic ExFT action (3.97) is the unique bosonic action that is invariant under both internal and external diffeomorphisms, up to overall rescalings and terms that vanish under the section condition. The terms in the Lagrangian (3.98) were first constructed to be invariant under internal diffeomorphisms and invariance under the external diffeomorphisms fixed all remaining coefficients. The uniqueness of the bosonic ExFT action is quite remarkable. Coming from supergravity one would expect that the symmetries fix the structure of the bosonic action but not the relative coefficients. Instead one would expect them to be determined by requiring the full action to be supersymmetric. Nonetheless, the unique bosonic ExFT action (3.97) admits a supersymmetric extension [26] — which may indicate a hidden relationship between supersymmetry and exceptional symmetry.

The fields transform under the covariantised external diffeomorphisms as the covariantisation of the standard Lie derivative, with all partial derivatives replaced by covariant derivatives. For the external vielbein (3.109) and for the scalar fields (3.110) this is the full transformation, but the transformations of the differential forms are further modified.

$$\delta_\xi E_\mu{}^\alpha = \xi^\nu\, \mathcal{D}_\nu E_\mu{}^\alpha + \mathcal{D}_\mu \xi^\nu\, E_\nu{}^\alpha \tag{3.109}$$

$$\delta_\xi M_{MN} = \xi^\mu\, \mathcal{D}_\mu M_{MN} \tag{3.110}$$

$$\Delta_\xi B_{\mu\nu M} = \frac{1}{16\kappa}\, \xi^\rho\, E\, \epsilon_{\mu\nu\rho\sigma\tau}\, \mathcal{F}^{\sigma\tau N}\, M_{MN} \tag{3.111}$$

$$\delta_\xi A_\mu^M = \xi^\nu\, \mathcal{F}_{\nu\mu}^M + M^{MN}\, g_{\mu\nu}\, \partial_N \xi^\nu \tag{3.112}$$

Naively one would expect $\Delta_\xi B_{\mu\nu M} = \xi^\rho\, \mathcal{H}_{\mu\nu\rho M}$ to be the transformation of the two-forms under external diffeomorphisms. But in order to realise the diffeomorphism symmetry of the action (3.97) off-shell one has to insert the two-form equation of motion (3.113) (i.e. the on-shell duality relation between the one- and two-forms) in the naive transformation to yield (3.111).

$$d^{PML}\,\partial_L\left(E\,M_{MN}\,\mathcal{F}^{\mu\nu N}+\kappa\,\epsilon^{\mu\nu\rho\sigma\tau}\,\mathcal{H}_{\rho\sigma\tau M}\right)=0 \tag{3.113}$$

The transformation of the one-forms is given by (3.112). The first term in (3.112) is the covariantisation of the expected transformation of a one-form under external diffeomorphisms.
The perhaps unexpected second term in (3.112) comes from a compensating Lorentz transformation. This term only exists for diffeomorphism parameters that depend on the internal coordinates. Because this term, in the transformation of the one-forms, depends on the external metric and the scalar fields it connects different terms in the Lagrangian. It is instructive to discuss the origin of this term in more detail and moreover the term will be relevant in the canonical formulation of the theory. To understand the origin of the second term in (3.112) we need to look at how the ExFT relates to eleven-dimensional supergravity. Here we follow the calculation presented in reference [25].

We begin with a Kaluza-Klein-like $11 = 5 + 6$ split of the eleven-dimensional indices. The eleven-dimensional curved index splits as $\hat{\mu} = (\mu, m)$ and the flat Lorentz index splits as $\hat{\alpha} = (\alpha, a)$.[13] In this split we can parametrise the eleven-dimensional vielbein as (3.114), where we have defined $\phi := \det(\phi_m{}^a)$ and $\gamma := -1/3$.

$$E_{\hat{\mu}}{}^{\hat{\alpha}}=\begin{pmatrix}\phi^\gamma\,E_\mu{}^\alpha & A_\mu^m\,\phi_m{}^a\\ 0 & \phi_m{}^a\end{pmatrix} \tag{3.114}$$

We can furthermore define $\phi^{mn} := \phi_a{}^m\,\phi^{an}$. We can interpret the $\phi_m{}^a$ as an internal vielbein and consequently the ϕ^{mn} as an internal metric. The A_μ^m are the Kaluza-Klein vectors. These fields are the "unextended" precursors to the scalar fields M^{MN} and one-forms A_μ^M of the ExFT. The eleven-dimensional Lorentz symmetry has to be partially gauge-fixed in order to achieve the upper-triangular form of (3.114).
The eleven-dimensional vielbein transforms under eleven-dimensional diffeomorphisms with parameters $\xi^{\hat{\nu}}$ and Lorentz transformations with parameters $\lambda^{\hat{\alpha}}{}_{\hat{\beta}}$ as (3.115).

$$\delta E_{\hat{\mu}}{}^{\hat{\alpha}}=\xi^{\hat{\nu}}\,\partial_{\hat{\nu}}E_{\hat{\mu}}{}^{\hat{\alpha}}+\partial_{\hat{\mu}}\xi^{\hat{\nu}}\,E_{\hat{\nu}}{}^{\hat{\alpha}}+\lambda^{\hat{\alpha}}{}_{\hat{\beta}}\,E_{\hat{\mu}}{}^{\hat{\beta}} \tag{3.115}$$

The transformation (3.115) implies the transformations of the components of the decomposition (3.114). Moreover the condition (3.114) on the vielbein, to be of upper-triangular form, implies that the vanishing components cannot transform non-trivially under the gauge transformations (3.115), i.e. $E_m{}^\alpha = 0 \;\Rightarrow\; \delta E_m{}^\alpha = 0$. As a consequence the condition $\delta E_m{}^\alpha = 0$ implies the partial gauge fixing (3.116) of the Lorentz gauge parameters.

$$\lambda^\alpha{}_b=-\phi^\gamma\,\phi_b{}^m\,\partial_m\xi^\nu\,e_\nu{}^\alpha \tag{3.116}$$

The structure of the Lorentz algebra implies further restriction on the Lorentz parameters (3.117).

$$\lambda^\alpha{}_b=-\delta_{ab}\,\eta^{\alpha\beta}\,\lambda^a{}_\beta \tag{3.117}$$

[13] The flat indices used here differ in nomenclature from those used in reference [25] in order to be consistent with notation used in the remainder of this thesis.

Looking at the transformation of the $E_\mu{}^a$ component of the eleven-dimensional vielbein and inserting the conditions (3.116) and (3.117) for the Lorentz gauge parameters we find that the transformation of A_μ^m is given by (3.118), which contains the term in question.

$$\delta_\xi A_\mu^m = \xi^\nu\, F_{\nu\mu}^M + \phi^{2\gamma}\, \phi^{mn}\, g_{\mu\nu}\, \partial_n \xi^\nu \tag{3.118}$$

In the extended exceptional geometry the transformation (3.118) of the Kaluza-Klein vectors of eleven-dimensional supergravity becomes the transformation (3.112) of the generalised vectors A_μ^M of ExFT. The correction term in (3.112) exists as a consequence of a compensating Lorentz transformation to keep the parametrisation (3.114) consistent. In particular the sign of the correction term in (3.112) is completely fixed — we will come back to this fact in chapter 6, when we discuss the canonical gauge transformations.

The final symmetry of the action (3.97) are external Lorentz transformations. The vielbein of ExFT $E_\mu{}^\alpha$ transforms under the five-dimensional external Lorentz transformations as (3.119).

$$\delta_\lambda E_\mu{}^\alpha = \lambda^\alpha{}_\beta\, E_\mu{}^\beta \tag{3.119}$$

In conclusion, the bosonic $\mathrm{E}_{6(6)}$ ExFT action (3.97) is invariant under external Lorentz transformations, external (covariantised) diffeomorphisms, internal generalised diffeomorphisms, two-form tensor gauge transformations and certain shift transformations.

3.5.5 Lagrangian gauge algebra

The parts of the gauge algebra that concern the covariantised external and internal generalised diffeomorphisms have been described in [25, 36], which we review in this section. We can write the commutator of two external (covariantised) diffeomorphisms as (3.120), which is the sum of an external and an internal diffeomorphism. The internal diffeomorphism appears here due to the covariantisation of the external diffeomorphisms. The dots in (3.120) indicate possible tensor gauge transformations of higher degree differential forms from the tensor hierarchy (e.g. three-form gauge transformations).

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_{12}} + \delta_{\Lambda_{12}} + \dots \tag{3.120}$$

The resulting external diffeomorphism parameter ξ_{12} is given by (3.121), which is the covariantised commutator of the original parameters. The internal diffeomorphism parameter Λ_{12}^M is given by (3.122). If we compare (3.122) to the transformation of the one-forms under external diffeomorphisms (3.112), we can see that this is a very similar expression.

$$\xi_{12}^\mu := \xi_2^\nu\, \mathcal{D}_\nu \xi_1^\mu - \xi_1^\nu\, \mathcal{D}_\nu \xi_2^\mu \tag{3.121}$$

$$\Lambda_{12}^M := \xi_2^\mu\, \xi_1^\nu\, \mathcal{F}_{\mu\nu}^M - 2\, M^{MN}\, g_{\mu\nu}\, \xi_{[2}^\mu\, \partial_N \xi_{1]}^\nu \tag{3.122}$$

The commutator of an external and an internal diffeomorphism is given by (3.123), which consists of an external diffeomorphism and a two-form tensor gauge transformation.

$$[\delta_\Lambda, \delta_\xi] = \delta_{\xi'} + \delta_{\Xi'} \tag{3.123}$$

The resulting external diffeomorphism parameter ξ'^μ in (3.123), is given by (3.124), which is the generalised Lie derivative of the original external parameter ξ^μ. The parameter Ξ' of the tensor gauge transformation is given by (3.125). We can compare

(3.125) to the gauge transformations and find that it is of the form of the transformation of the one-forms under external diffeomorphisms (3.112) inserted in a generalised diffeomorphism acting on the two-forms (3.107).

$$\xi'^{\mu} := -\Lambda^M \, \partial_M \xi^\mu \tag{3.124}$$
$$\Xi'_{\mu M} := -d_{MNK} \, \left(\xi^\nu \, \mathcal{F}^N_{\nu\mu} + M^{KL} \, g_{\mu\nu} \, \partial_L \xi^\nu\right) \, \Lambda^K \tag{3.125}$$

The generalised diffeomorphisms form the subalgebra (3.126) of the gauge algebra. Because the generalised diffeomorphism transformations can be written as the generalised Lie derivative $\delta_\Lambda = \mathbb{L}_\Lambda$ the algebra (3.126) is equivalent to the commutator of the generalised Lie derivatives (3.79).

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] = \delta_{\Lambda_{12}} \tag{3.126}$$

The resulting gauge parameter Λ_{12} is, by definition, given by the E-bracket (3.127) of the original parameters.

$$\Lambda^M_{12} := [\Lambda_2, \Lambda_1]^M_{\mathrm{E}} \tag{3.127}$$

The generalised diffeomorphism algebra (3.126) closes only up to terms that vanish under the section condition (3.72), as was discussed in section 3.5.2.

3.5.6 Solutions of the ExFT section condition

Now that we have discussed the field content, the action, the gauge transformations and the gauge algebra of the bosonic $\mathrm{E}_{6(6)}$ exceptional field theory we can discuss how the ExFT relates back to supergravity. We constructed the ExFT from a Kaluza-Klein-like split of eleven-dimensional supergravity, without truncating any degrees of freedom, and then transformed the internal six-dimensional Riemannian geometry into a 27-dimensional extended generalised exceptional geometry. The extended generalised exceptional geometry is what allowed us to write the action of ExFT in an $\mathrm{E}_{6(6)}$ invariant form, but its consistency required the introduction of the section condition (3.72).
The section condition should be seen, not as a condition on the fields, but instead as a condition on the 27 extended coordinates Y^M. We can solve the section condition (3.72) by selecting a suitable subset of the extended coordinates, thereby undoing the construction and breaking the $\mathrm{E}_{6(6)}$ symmetry to a subgroup thereof. Solutions of the $\mathrm{E}_{6(6)}$ section condition contain at most six of the 27 coordinates, but solutions with fewer coordinates exist.

The solution of the section condition that leads back to eleven-dimensional supergravity is found by considering the embedding (3.128) of the subgroup GL(6) into $\mathrm{E}_{6(6)}$.

$$\mathrm{GL}(6) = \mathrm{SL}(6) \times \mathrm{GL}(1) \subset \mathrm{E}_{6(6)} \tag{3.128}$$

The fundamental 27 and the adjoint 78 representations of $\mathrm{E}_{6(6)}$ decompose into the direct sum of representations of GL(6) according to (3.129) and (3.130) respectively, where the index indicates the weight under the GL(1).

$$27 \to 6_{+1} \oplus 15'_0 \oplus 6_{-1} \tag{3.129}$$
$$78 \to 1_{-2} \oplus 20_{-1} \oplus (1 \oplus 35)_0 \oplus 20_{+1} \oplus 1_{+2} \tag{3.130}$$

The internal coordinates Y^M decompose under the embedding (3.129) as (3.131), with the GL(6) indices $m, \overline{m}, n, \overline{n} = 1, \ldots, 6$ — the overline is used here to indicate the difference in GL(1) weight. The $y_{mn} = y_{[mn]}$ are antisymmetric and therefore represent 15 independent coordinates.

$$Y^M \to (y^m, y_{mn}, y^{\overline{m}}) \tag{3.131}$$

To verify that the section condition can be solved in this embedding the section condition (3.72) itself has to be decomposed too. The only non-vanishing components of the invariant symbol d^{MNK} are given by (3.132) [25].

$$d^{m\,\overline{n}}{}_{kl} = \frac{1}{\sqrt{5}}\,\delta^m_{[k}\,\delta^{\overline{n}}_{l]} \quad , \quad d_{mn\,kl\,pq} = \frac{1}{4\sqrt{5}}\,\epsilon_{mnklpq} \tag{3.132}$$

Now that we know how to write the section condition in the GL(6) decomposition (3.129), we can identify a suitable subset of the extended coordinates that solves the section condition. We find that if we keep only the 6_{+1} coordinates y^m from the 27 extended coordinates Y^M (3.131) and therefore only the internal derivatives ∂_m, this solves the section condition. The 21 remaining coordinates are dropped, i.e. $\partial^{mn} = 0$ and $\partial_{\overline{m}} = 0$. In this solution of the section condition the full set of coordinates are then the $11 = 5 + 6$ external and internal coordinates (x^μ, y^m). We should now decompose all objects according to (3.129) and only keep the 6_{+1} components. Furthermore the Hodge-dualisations that were necessary in the construction of the ExFT have to be undone. The result is then that one recovers the full structure of eleven-dimensional supergravity — written in the Kaluza-Klein-like $5+6$ split.

When the above GL(6) solution to the section condition is chosen one can view the ExFT as a true, though possibly unusual, rewriting of eleven-dimensional supergravity, because no degrees of freedom were truncated and when the section condition is applied none were added. To realise the $E_{6(6)}$ symmetry one has to make use of the extended exceptional geometry however, which suggests that the $E_{6(6)}$ symmetry does not exist in the eleven-dimensional theory in the usual sense. Conversely this may be seen as pointing towards a more fundamental interpretation of the exceptional geometry.

A further advantage of the ExFT is that there is more than one solution to the section condition and in this sense the ExFT is a unified description of several supergravity theories — as illustrated by figure 3.2. Instead of GL(6) one can choose to embed $\mathrm{GL}(5)\times\mathrm{SL}(2) \subset E_{6(6)}$. When keeping only the $(5,1)_{+4}$ components of the $\mathrm{GL}(5)\times\mathrm{SL}(2)$ decomposition one finds another inequivalent but consistent solution to the section condition. The supergravity theory that is reached by this solution is the ten-dimensional type IIB supergravity [36, 60, 61]. This fact is all the more interesting because the type IIB theory cannot be reached by toroidal compactification from eleven-dimensions. Furthermore one can choose to embed a different GL(5), which is contained in the GL(6) that leads to eleven-dimensional supergravity, keeping five of the 6_{+1} coordinates then leads to ten-dimensional type IIA supergravity, which can be reached by a circle compactification from eleven-dimensions.
For the $E_{7(7)}$ ExFT it has moreover been shown that it is possible to introduce consistent deformations of the generalised Lie derivative which makes it possible to arrive at the massive type IIA supergravity via the section condition [168]. Therefore all maximal supergravity theories in ten and eleven dimensions can be reached from ExFT.

Furthermore the section condition can be trivially solved by discarding all internal

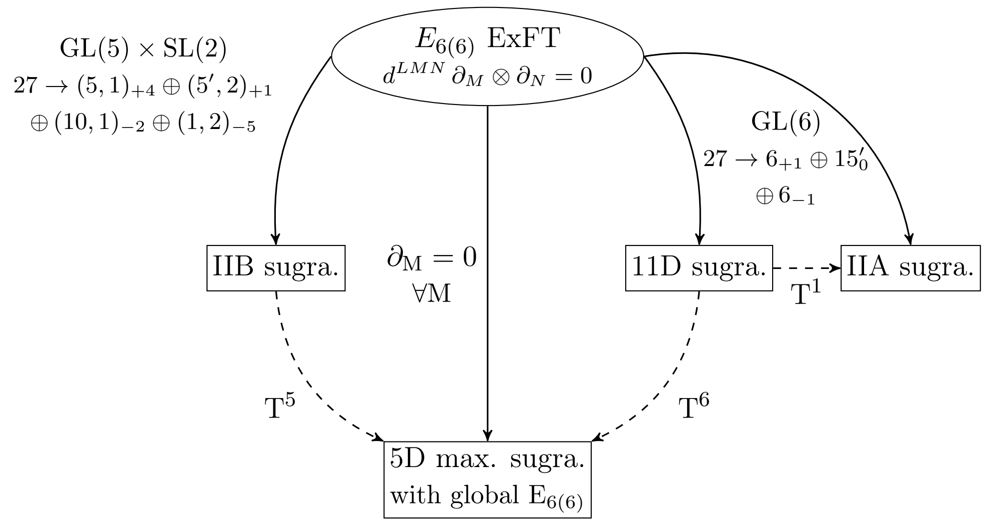


FIGURE 3.2: The embedding of the eleven-dimensional, type IIA, type IIB and ungauged maximal five-dimensional supergravity theories in the $\mathrm{E}_{6(6)}$ ExFT. The solid lines indicated the $\mathrm{GL}(5) \times \mathrm{SL}(2)$, $\mathrm{GL}(6)$ and trivial solutions of the section condition. Dashed lines indicate toroidal compactifications on m-tori T^m. Adapted from a similar figure in reference [25].

coordinates $\partial_M = 0\ \forall M$ and therefore only the five external coordinates survive — this can be seen as a reduction on an internal T^{27} torus. The supergravity theory that is reached in this solution of the section condition is the manifestly $\mathrm{E}_{6(6)}$ invariant form of the ungauged maximal five-dimensional supergravity, which was first described in [74]. This is the only solution to the section condition that does not break the $\mathrm{E}_{6(6)}$ symmetry, however it only survives as a global symmetry in five dimensions. As discussed in section 2.3.2, this five-dimensional theory can also be reached by a toroidal reduction of eleven-dimensional supergravity on a six-torus. In contrast to the toroidal compactification the trivial solution of the ExFT leads directly to the manifestly $\mathrm{E}_{6(6)}$ invariant form, without requiring any further changes.
The canonical formulation and analysis of this five-dimensional theory are carried out in chapter 5.

It is moreover possible to partially solve the section condition for the NSNS sector to arrive at DFT coupled to additional RR-fields, as has been shown in [169] for the $\mathrm{E}_{4(4)}$ theory [141]. Due to the embedding of $\mathrm{E}_{n-1(n-1)} \subset \mathrm{E}_{n(n)}\ \forall n$ one would suspect that it should also be possible to reach any of the $\mathrm{E}_{m(m)}$ ExFTs from any $\mathrm{E}_{n(n)}$ ExFT if $m < n$. This question has been examined in [170] for some cases, but not all aspects about these reductions are understood and questions about some aspects of the reductions, in particular concerning the tensor hierarchy and the topological terms, remain.

3.5.7 Applications of exceptional field theory

Since its development, almost a decade ago, exceptional field theory has grown into a very active field with a diverse range of topics and applications and here we want to mention some examples of these applications — this list is however far from being comprehensive. An overview of the applications of ExFT can also be found in [103].

Perhaps we should judge the main "application" of ExFT to be the fact that its $\mathrm{E}_{n(n)}$ invariant action in particular encodes the dynamics of eleven-dimensional supergravity, as was discussed in section 3.5.6. A further immediate application is that the $\mathrm{GL}(5)\times\mathrm{SL}(2) \subset \mathrm{E}_{6(6)}$ solution of the section condition leads to an off-shell action for the type IIB supergravity [25, 36].

Many applications of ExFT are concerned with the construction of consistent reductions, gaugings and the generalised Scherk-Schwarz reduction procedure. Scherk-Schwarz reductions [65] are a generalisation of the toroidal compactification procedure to more general manifolds via a factorisation ansatz for the coordinate dependence. The generalised Scherk-Schwarz reduction procedure has been developed in [155, 171–179] and similarly assumes a (twisted) factorisation of the coordinate dependence for the fields of ExFT, e.g. $M_{MN}(x, Y) = U^A{}_M(Y)\, U^B{}_N(Y)\, \tilde{M}_{AB}(x)$ for the scalar fields, where $U^A{}_M(Y) \in \mathrm{E}_{6(6)}$ is the twist matrix, which has to satisfy certain (Scherk-Schwarz) consistency conditions. Using a generalised Scherk-Schwarz reduction, the ExFT can be reduced on a twisted torus to arrive at the five-dimensional SO(6) gauged maximal supergravity theory [180, 181], which was first constructed in [182]. Moreover one can use this to prove the consistency of the equivalent reduction of the type IIB supergravity on an $\mathrm{AdS}_5\times\mathrm{S}^5$ manifold, which leads to the same gauged supergravity [180, 181]. In this sense the generalised Scherk-Schwarz reductions of ExFT can be used to make statements about reductions of supergravity. The twisted tori generalised Scherk-Schwarz reductions of the ExFT can also lead to the $\mathrm{SO}(p, q)$ and $\mathrm{CSO}(p, q, r)$ gauged maximal supergravity theories that can otherwise be constructed from the type IIB theory by reduction on warped hyperboloids [183, 184]. One can furthermore identify the Kaluza-Klein mass spectra of the reduced theory [179]. Reductions with less than maximal supersymmetry have also been constructed [185–189].

Another topic concerns the study of solutions that are "non-geometric", in the sense that one cannot define a metric globally, instead one has to make use of duality transformations to connect local patches of coordinates. These geometries are hence not solutions of ordinary supergravity theories, which are not duality-covariant. In the case of U-folds and T-folds the transformations are the U- and T-duality transformations [190, 191] and similarly one can study O-folds (or generalised orbifolds) where the extended coordinates are related by transformations $Y^M \sim Z^M{}_N\, Y^N$ where $Z^M{}_N$ is an element of a discrete subgroup of $\mathrm{E}_{n(n)}$ [192]. Furthermore reductions on non-geometric spaces could be relevant to finding de Sitter solutions of supergravity [193].

One can moreover take advantage of the duality covariance of ExFT (or DFT) to construct duality invariant amplitudes [33–35] and α' (string length scale) corrections, see e.g. [194–197]. An exceptional sigma-model, which can be reduced to the doubled sigma model, has also been constructed [198].

Another direction of research is the formalisation of the structure of ExFT and tensor hierarchy in terms of L_∞ algebras [199–202].

3.5.8 The explicit non-integral form of the topological term $\mathcal{L}_{\mathrm{top}}$

This section is based on the publication [41]. In section 3.5.3 the complete (bosonic) Lagrangian of the $\mathrm{E}_{6(6)}$ ExFT was presented, as published in [3, 25]. The terms of the Lagrangian (3.98) were stated explicitly, with the exception of the topological term $\mathcal{L}_{\mathrm{top}}$, which is instead given as S_{top} (3.105), which is a manifestly invariant $(6+27)$-dimensional integral over an exact six-form. Reference [25] furthermore states the general variation $\delta\mathcal{L}_{\mathrm{top}}$ as (3.133).

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{top}} = \kappa\,\epsilon^{\mu\nu\rho\sigma\tau}(&\frac{3}{4}\,d_{MNK}\,\mathcal{F}_{\mu\nu}^{M}\,\mathcal{F}_{\rho\sigma}^{N}\,\delta A_{\tau}^{K} + 5\,d^{MNK}\,d_{KQP}\partial_N\mathcal{H}_{\mu\nu\rho M}A_{\sigma}^{P}\delta A_{\tau}^{Q} \\ &+ 5\,d^{MNK}\,\partial_N\mathcal{H}_{\mu\nu\rho M}\delta B_{\sigma\tau K})\end{aligned} \tag{3.133}$$

From the point of view of the canonical analysis of the theory the general variation (3.133) would be sufficient for the calculation of the contributions to the canonical momenta of the one- and two-forms. Where the integral form of the topological term (3.105) is problematic is the Legendre transformation of the Lagrangian, because the topological term mixes with the Legendre transformation terms. Hence we need to construct the explicit non-integral form of the topological Lagrangian $\mathcal{L}_{\mathrm{top}}$, which is not manifestly gauge invariant, in order to carry out the Legendre transformation.

In the previous sections we have seen that the structure of the $\mathrm{E}_{6(6)}$ ExFT is often analogous to that of gauged five-dimensional maximal supergravity [84]. The analogies between these theories and in particular the similarity of the topological terms have also been pointed out in [25]. We can use this similarity to find the explicit form of the topological term $\mathcal{L}_{\mathrm{top}}$. First we construct an ansatz, with general coefficients for each term, based on the topological term of gauged supergravity, cf. equation (3.11) of [84], which is known explicitly, but depends on the "structure constants" (2.29) of the gauging. Then we fix the coefficients of the ansatz by comparing the general variation of the ansatz to the variation $\delta\mathcal{L}_{\mathrm{top}}$ given in (3.133).

In section 2.3.4 we have seen that the "structure constants" $X_{MN}{}^{P}$, which appear in the topological term of gauged supergravity can be split into a symmetric and an antisymmetric part (3.134) (cf. equation (2.29)).

$$X_{MN}{}^{P} = X_{(MN)}{}^{P} + X_{[MN]}{}^{P} \tag{3.134}$$

In order to write down a suitable ansatz for the ExFT topological term these $X_{MN}{}^{P}$ have to be replaced by objects from the ExFT itself. The split of (3.134) into a symmetric and an antisymmetric part is analogous to the relation (3.82) of the Dorfman bracket of ExFT. Moreover the covariant one-form field strength of gauged supergravity (2.33) can be compared to the analogous expression (3.90) in ExFT. We can compare further structures, but these comparisons all point to the following identifications. The antisymmetric part of the structure constants should be analogous to the E-bracket $X_{[MN]}{}^{K}\,A_{\mu}^{M}\,A_{\nu}^{N} \sim [A_{\mu}, A_{\nu}]_{E}^{K}$ and the projector $Z^{KL} \sim d^{KLM}\,\partial_M$ can be seen as being analogous to the symmetric part. Because general coefficients are used in the ansatz the precise relations between these objects is irrelevant. These identifications can now be inserted in the topological term of gauged supergravity, which is given in (3.11) of [84], to yield the desired ansatz.

Next we compute the general variation of this ansatz and compare it to the variation (3.133) to fix all coefficients and arrive at the topological term of the $\mathrm{E}_{6(6)}$ ExFT.

What we find is that the topological term given by (3.135) has the same general variation $\delta\mathcal{L}_{\text{top}}$ as given by (3.133).

$$\begin{aligned}
\mathcal{L}_{\text{top}} = & -\frac{5\kappa}{2}\epsilon^{\mu\nu\rho\sigma\tau}\ d^{MNR}\ \partial_R B_{\mu\nu M} \\
& \times\left[3\mathcal{D}_\rho B_{\sigma\tau N} - 6 d_{NKL}\, A_\rho^K \left(\partial_\sigma A_\tau^L - \frac{1}{3}[A_\sigma, A_\tau]_{\text{E}}^L\right)\right] \\
& + \kappa\ \epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ \partial_\nu A_\rho^M\ \partial_\sigma A_\tau^P \\
& - \frac{3\kappa}{4}\epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ [A_\nu, A_\rho]_E^M\ \partial_\sigma A_\tau^P \\
& + \frac{3\kappa}{20}\epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ [A_\nu, A_\rho]_E^M\ [A_\sigma, A_\tau]_E^P
\end{aligned} \tag{3.135}$$

The topological term (3.135) can be rewritten in the slightly more covariant form (3.136) using the definition of the covariant two-form field strength (3.95). We will however only make use of the more explicit form (3.135) in the remainder of this thesis.

$$\begin{aligned}
\mathcal{L}_{\text{top}} = & -\frac{5\kappa}{2}\epsilon^{\mu\nu\rho\sigma\tau}\ d^{MNR}\ \partial_R B_{\mu\nu M} \\
& \times\left[\mathcal{H}_{\rho\sigma\tau N} - 3 d_{NKL}\, A_\rho^K \left(\partial_\sigma A_\tau^L - \frac{1}{3}[A_\sigma, A_\tau]_{\text{E}}^L\right)\right] \\
& + \kappa\ \epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ \partial_\nu A_\rho^M\ \partial_\sigma A_\tau^P \\
& - \frac{3\kappa}{4}\epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ [A_\nu, A_\rho]_E^M\ \partial_\sigma A_\tau^P \\
& + \frac{3\kappa}{20}\epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A_\mu^N\ [A_\nu, A_\rho]_E^M\ [A_\sigma, A_\tau]_E^P
\end{aligned} \tag{3.136}$$

Chapter 4

Constrained Hamiltonian systems and canonical case studies

In consideration of the importance of the Hamiltonian formalism to the content of this thesis, we review the foundational theory of the Hamiltonian formalism in section 4.1. We use the terms Hamiltonian formalism and canonical formalism interchangeably.[1] A complete introduction to the Hamiltonian formalism is far beyond the scope of this brief introductory section and we review only the basic definitions and statements that are needed in the following chapters — necessarily omitting many details. Some of the standard references that give a very detailed account of the intricacies of the canonical formalism are [27, 28].
Following the general discussion of the Hamiltonian formalism in section 4.1 we introduce the Arnowitt-Deser-Misner (ADM) [203] formulation of general relativity in section 4.2, focusing on the space-time decomposition of the degrees of freedom. In section 4.3 the canonical formulation of the $\mathrm{SL}(n)/\mathrm{SO}(n)$ scalar coset sigma model is investigated, with a focus on the canonical treatment of the coset constraints. The insights that we derive from this model theory are valuable when dealing with the much more complicated $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset of the scalar fields of $\mathrm{E}_{6(6)}$ ExFT. In section 4.4 we study the canonical formulation of a certain model theory of topological two-forms in $5+27$ dimensions, based on the topological kinetic term of the two-forms in the $\mathrm{E}_{6(6)}$ ExFT. This topological toy model illustrates some of the main challenges concerning the second class nature of the two-form constraints.

Section 4.1 is based on parts of the publication [40], which reviews some of the contents of [27, 28]. Section 4.2 is based on parts of the publication [40], which reviews [203]. Section 4.3 is based on parts of the publication [40] and section 4.4 is based on parts of the publication [41].

4.1 Foundational theory of the Hamiltonian formalism

The study of constrained Hamiltonian systems encompasses the study of gauge (field) theories. Gauge theories are ubiquitous in physics, because the introduction of auxiliary frames of reference is able to simplify the description of the true physical degrees of freedom by increasing the overall symmetry. The presence of these auxiliary reference frames entails the freedom to make changes to the reference frames, which

[1] Sometimes we may distinguish between the canonical formulation and the canonical analysis, though these terms are not always distinguished in other works. We consider the canonical formulation to be the study of the canonical momenta, the construction of the canonical Hamiltonian and the construction of the complete and consistent set of canonical constraints and consider the canonical analysis to be the study of the transformations generated by the canonical constraints, the algebra of the canonical constraints and in general the properties of the constraints and the Hamiltonian.

in turn leads to transformations of the variables (fields) of the theory that we call gauge transformations. Only the true physical degrees of freedom, the observables, are invariant under these gauge transformations. The transformations of the auxiliary variables can moreover be time dependent and thus, for a gauge theory, a given set of initial conditions does not lead to a unique time evolution. This arbitrariness in the time evolution implies that not all of the variables, that are used to describe the theory, are independent — this is expected since we know that the auxiliary reference frames are not truly physically meaningful. The study of the constraints that relate the variables of the theory and the transformations that they generate are a central part of the Hamiltonian formalism.
The Hamiltonian formalism may be seen as being more fundamental than the Lagrangian formalism because it is able to find the most general time evolution where the full gauge symmetry is manifest. Furthermore the Hamiltonian formalism provides an algorithmic framework with which we can systematically identify the gauge symmetries, constraints, observables, etc. The Lagrangian formalism on the other hand also has its advantages, e.g. when wanting to write down a theory that obeys the principle of relativity. And it is normally the case that one starts the Hamiltonian formalism coming from the Lagrangian description of the theory and this is were we begin too.

In the Lagrangian formalism a theory is written as an action S and the principle of stationary action $\delta S = 0$ is used to determine the equations of motion and solutions of the theory. The action can be written as a functional (4.1), by taking a d-dimensional space-time integral of the Lagrangian density $\mathcal{L}(Q^n, \dot{Q}^n)$, which in turn depends on a number of fields $Q^n(x)$ $(n = 1, \ldots, N)$ and their time derivatives $\dot{Q}^n(x)$.

$$S = \int d^d x \; \mathcal{L}(Q^n, \dot{Q}^n) \tag{4.1}$$

We now introduce the canonical momenta $\Pi_n(Q)$ as defined by the functional derivative of the Lagrangian density to the time derivatives of the fields (4.2). Throughout this thesis we use the notation that a capital $\Pi(Q)$ — with appropriate indices — signifies the canonical momentum conjugate to the field Q. Sometimes the specification of the field is omitted if it is clear which field is meant.

$$\Pi_n(Q) := \frac{\delta \mathcal{L}}{\delta \dot{Q}^n} \tag{4.2}$$

The canonical momenta are generally treated on an equal footing to the fields in the Hamiltonian formalism. Therefore we define the Hamiltonian density $\mathcal{H}(Q^n, \Pi_n(Q))$ by the Legendre transformation of the Lagrangian density (4.3) that eliminates the time derivatives of the fields as variables and replaces them with the canonical momenta. We can likewise rewrite the action as (4.4).[2]

$$\mathcal{H}(Q^n, \Pi_n(Q)) := \dot{Q}^n \, \Pi_n(Q) - \mathcal{L}(Q^n, \dot{Q}^n) \tag{4.3}$$

$$S = \int d^d x \; \Big(\dot{Q}^n \, \Pi_n(Q) - \mathcal{H}(Q^n, \Pi_n(Q)) \Big) \tag{4.4}$$

[2]The terms Lagrangian density and Lagrangian, as well as Hamiltonian density and Hamiltonian are used interchangeably in this thesis — in the true Lagrangian or Hamiltonian the dependence on the spatial coordinates is integrated out.

The Hamiltonian $\mathcal{H}(Q^n, \Pi_n(Q))$ only depends on the canonical coordinates (also called Darboux coordinates) $(Q^n, \Pi_n(Q))$ and we call the space that they parametrise phase space. As explained above, in a gauge theory not all of the $2N$ coordinates of phase space are physically meaningful and there are constraints that relate some of the canonical coordinates. In particular we find that some of the canonical momenta yield equations of the form $\mathcal{H}_m(Q^n, \Pi_n(Q)) = 0 \;\; (m = 1, \ldots, M)$ that only depend on the canonical coordinates and not on their time derivatives and we call these primary constraints. We ought not to confuse the constraints $\mathcal{H}_m$ with the canonical Hamiltonian $\mathcal{H}$, but we will see that they are indeed closely related — this is particularly true for the case of generally covariant theories, such as general relativity and we will see that in this case the Hamiltonian consists entirely of constraints.

Since the primary constraints follow directly from the definition of the canonical momenta, we have not used the equations of motion to find them and they cannot place any restriction on the dynamics. The simplest primary constraint is a vanishing canonical momentum $\mathcal{H}_m(Q^n, \Pi_n(Q)) = \Pi_m(Q) = 0$. This type of constraint is quite common and appears when there are no time derivatives on a field in the Lagrangian. We refer to constraints of this type as shift type primary constraints because they generate shift transformations on the conjugate field variables. Similarly when there are fields whose kinetic term is linear in time derivatives, we find constraints that directly relate field and momentum $\Pi_m(Q) \sim Q_m$ (cf. Dirac spinors) and therefore also generate shift transformations of the field. This will become clear once we have seen how gauge transformations are generated by the constraints.

Because of the constraints, the Hamiltonian is only well defined on the hypersurface in phase space that is defined by the set of primary constraints $\mathcal{H}_m(Q^n, \Pi_n(Q)) = 0 \; \forall m$. In particular this implies that we can add a general linear combination of the primary constraints to the Hamiltonian because the linear combination is by definition always vanishing on the primary constraint surface. This way we can extend the Hamiltonian arbitrarily away from the primary constraint surface as the coefficient functions are arbitrary and we call this function (4.5) the total Hamiltonian $\mathcal{H}_T$.

$$\mathcal{H}_T := \mathcal{H} + u^m \; \mathcal{H}_m \tag{4.5}$$

Another central object in the Hamiltonian formalism is the Poisson bracket, which is closely related to the symplectic structure of phase space. The Poisson bracket is a binary operation and using the coordinates of phase space we can define it as (4.6), where F, G are phase space functions.[3]

$$\{F, G\} := \frac{\delta F}{\delta Q^n} \frac{\delta G}{\delta \Pi_n} - \frac{\delta G}{\delta Q^n} \frac{\delta F}{\delta \Pi_n} \tag{4.6}$$

One can easily verify that the Poisson bracket is antisymmetric in its arguments, bilinear, obeys the Leibniz rule and satisfies the Jacobi identity. Regarding the canonical coordinates the only non-vanishing Poisson bracket relations are the fundamental (equal time) Poisson brackets (4.7).

$$\{Q^n(x), \Pi_{\tilde{n}}(\tilde{x})\} = \delta^n_{\tilde{n}} \, \delta^{(d-1)}(x - \tilde{x}) \tag{4.7}$$

[3]There is a hidden space-time integral over the coordinate dependence of the derivatives in the definition since we are working with functional derivatives.

The time evolution of a phase space function F can be written as (4.8). At this point this is the most general time evolution we can write down. Note that we can clearly see the arbitrariness in the time evolution in (4.8).

$$\dot{F} = \{F, \mathcal{H}\} + u^m \{F, \mathcal{H}_m\} \tag{4.8}$$

For the formalism to be consistent the primary constraints need to be constant in time and the equations (4.9) need to hold.

$$\dot{\mathcal{H}}_m = \{\mathcal{H}_m, \mathcal{H}\} + u^{\tilde{m}} \{\mathcal{H}_m, \mathcal{H}_{\tilde{m}}\} \stackrel{!}{=} 0 \tag{4.9}$$

If a consistency condition (4.9) is independent of the arbitrary parameters u^m and independent of the primary constraints we call it a secondary constraint $\mathcal{H}_k$. The secondary constraints also need to be conserved and may yield further tertiary constraints. We iterate this procedure until it terminates — for reasonable physical theories termination is guaranteed.

Because we have made use of the equations of motion (4.8) to find the secondary constraints they can restrict the dynamics of the theory. This is the only difference between the primary and secondary constraints and we will not need to distinguish them further. From now on we will simply refer to the complete set of constraints as $\mathcal{H}_j,\ (j = 1, \ldots, J)$.

In this thesis we make frequent use of the concept of smeared or integrated constraints, in order to avoid writing derivatives of Dirac delta distributions. The smeared constraints $\mathcal{H}_{\text{Constr.}}[\lambda]$ are defined as the spatial integral of the constraints $\mathcal{H}_j$ fully contracted with an appropriate tensor of test functions $\lambda^j(x)$ (4.10).

$$\mathcal{H}_{\text{Constr.}}[\lambda] := \int d^{(d-1)}x\ \lambda^j\ \mathcal{H}_j \tag{4.10}$$

Another useful notational tool is the weak equality sign " $\approx$ ", which indicates that the equality holds on the phase space hypersurface defined by the complete set of constraints $\mathcal{H}_j = 0\ \forall j$. The equality that holds throughout phase space is by contrast also called strong equality. One should note that just because a function is weakly zero does not imply that Poisson brackets of this function are vanishing too (cf. $\det(M) - 1 \approx 0$ in section 4.3).

Having gone through the consistency algorithm with the consistency condition (4.9) we should have a complete set of canonical constraints. The u^m dependent equations that we have so far ignored pose a set of differential equations for the — a priori — arbitrary phase space coefficients u^m. The general solution of these differential equations can be written in terms of the linearly independent homogeneous solutions $V_a^m\ (a = 1, \ldots, A)$ and a particular solution U^m (4.11). The remaining v^a coefficients of the homogeneous solution are the truly arbitrary part of u^m. The time evolution $\dot{F} \approx \{F, \mathcal{H}_T\}$ generated by the total Hamiltonian is equivalent to the Lagrangian time evolution.

$$u^m = U^m + v^a\ V_a^m \tag{4.11}$$

While the distinction between primary and other constraints is not relevant after the construction of the complete set of constraints, there is an important property of phase space functions that we call the class. A phase space function F — this includes the constraints — is called a first class function if it Poisson-commutes weakly with every canonical constraint (4.12). Conversely if there exists a constraint with which the function does not weakly Poisson-commute then we call it a second class function (4.13). Clearly if all the constraints are first class, the algebra that the constraints form under the Poisson bracket closes.

$$\begin{aligned} F \text{ first class} \quad &\Leftrightarrow \forall j : \{F, \mathcal{H}_j\} \approx 0 && (4.12) \\ F \text{ second class} \quad &\Leftrightarrow \exists j : \{F, \mathcal{H}_j\} \not\approx 0 && (4.13) \end{aligned}$$

First class constraints generally have the interpretation of generators of gauge transformations. For primary constraints this is guaranteed, for first class secondary constraints the Dirac conjecture states that this should be true for physical theories.[4]

The infinitesimal gauge transformation δF of a phase space function F is generated by the right action of the Poisson bracket with the constraints $\mathcal{H}_a := \mathcal{H}_m\, V_a^m$ and the arbitrary gauge parameters v^a (4.14).

$$\delta F = \{F, \mathcal{H}_a\}\, v^a \tag{4.14}$$

The total Hamiltonian only contains the primary first class constraints, but we can improve on this by adding the complete set of first class constraints $\mathcal{H}_a^{(fc)}$ and thus define the extended Hamiltonian (4.15).

$$\mathcal{H}_E := \mathcal{H}_T + w^a\, \mathcal{H}_a^{(fc)} \tag{4.15}$$

The time evolution generated by the extended Hamiltonian $\mathcal{H}_E$ (4.16) is the most general time evolution we can write down since it contains the full gauge freedom, in this sense it transcends the Lagrangian formalism. By definition observables are invariant under gauge transformations and hence their extended time evolution is equivalent to the canonical and total time evolution $\mathcal{H}_E \Leftrightarrow \mathcal{H}_T \Leftrightarrow \mathcal{H}$.

$$\dot{F} \approx \{F, \mathcal{H}_E\}, \quad \mathcal{H}_j \approx 0 \tag{4.16}$$

Generally covariant theories — this includes in particular general relativity and supergravity — have Hamiltonians that vanish weakly $\mathcal{H} \approx 0$. The interpretation of this is that there are only gauge transformations and time evolution itself is just a gauge transformation generated by the Hamilton constraint.

If second class constraints are present they have to be handled separately. We can think of second class constraints in terms of first class constraints that have been gauge fixed, meaning that some condition has been imposed to remove the arbitrariness and hence the do not generate gauge transformation. Conversely we can gauge fix first class constraints and turn them into second class constraints.

Taking the set of second class constraints to be $\mathcal{H}_a^{(sc)}$ we can define the matrix given by their Poisson bracket relations amongst each other $M_{ab} := \{\mathcal{H}_a^{(sc)}, \mathcal{H}_b^{(sc)}\}$. The determinant of this matrix is non-zero if the constraints are indeed all second class and we can invert it by requiring $M_{ab}M^{bc} = \delta_a^c$ for an appropriate identity on the

[4]Counterexamples to the Dirac conjecture exist, as discussed in [28], however they tend to be rather special theories e.g. $\mathcal{L} = \frac{1}{2}\, e^y\, \dot{x}^2$.

constraints. The Dirac bracket is then defined as (4.17).

$$\{F,G\}_{\mathrm{DB}} := \{F,G\} - \{F,\mathcal{H}_a^{(sc)}\}\, M^{ab}\, \{\mathcal{H}_b^{(sc)},G\} \tag{4.17}$$

The Dirac bracket is antisymmetric, obeys the Leibniz rule and the Jacobi identity, but it has further properties that make it very useful. If we take the Dirac bracket of any second class constraint with an arbitrary function we find that it vanishes by construction $\{\mathcal{H}_a^{(sc)},F\}_{\mathrm{DB}} = 0$. The Dirac bracket of any first class constraint with an arbitrary function weakly reduces to the same Poisson bracket $\{\mathcal{H}_a^{(fc)},F\}_{\mathrm{DB}} \approx \{\mathcal{H}_a^{(fc)},F\}$ as can easily be seen from the definition. It follows that the algebra that the canonical constraints form under the Dirac bracket necessarily closes.

To determine the number of physical degrees of freedom of a theory — or equivalently the number of independent canonical coordinates — we only need to know the number of first class and second class constraints. A first class constraint is equivalent to two second class constraints because we can think of it as a gauge fixing condition and a second class constraint. We count the number of independent canonical coordinates according to (4.18) [28]. The number of physical degrees of freedom is half of the number of independent canonical coordinates since we are not counting the momenta in this case.

$$\binom{\text{Number of indep.}}{\text{canonical coord.}} = \binom{\text{Number of}}{\text{canonical coord.}} - 2\cdot\binom{\text{Number of}}{1^{\text{st}}\text{ cl. constr.}} - \binom{\text{Number of}}{2^{\text{nd}}\text{ cl. constr.}} \tag{4.18}$$

4.2 Arnowitt-Deser-Misner (ADM) formulation of general relativity

In this section we take a first look at the canonical Arnowitt-Deser-Misner (ADM) formulation of general relativity, which was first published in 1959 [203, 204]. The ADM formulation of general relativity takes a globally hyperbolic d-dimensional Lorentzian manifold $\mathcal{M}_d$ and foliates (or slices) the manifold into spatial hypersurfaces Σ_t labelled by a time coordinate $\mathcal{M}_d = \Sigma_t \times \mathbb{R}$. The condition of global hyperbolicity in particular guarantees that there are no closed causal curves and allows for this foliation to exist. In this section we focus on the structure of the ADM decomposition of the fields under this foliation of space-time, because this is needed in the following chapters. We omit some topics in this section, in particular the detailed discussion of the canonical momenta, the Hamiltonian, the constraints and their gauge symmetries and constraint algebra, whose discussion can be found in chapter 5, where we discuss the five-dimensional case, which is in substance identical to the discussion in a general dimensions. The topic is also covered in detail in references [13, 54, 55, 203–205]. The letters for the different indices used in this section are listed in table 4.1.

Type of index	Dimension	Letters used
Curved (external)	d	$\mu,\nu,\rho,\sigma,\tau,\ldots$
Curved (time)	1	t
Curved (external spatial)	$d-1$	$k,l,m,n,o,p,q,r,s,u,\ldots,x,y,z$
Flat (external)	d	$\alpha,\beta,\gamma,\delta,\ldots$
Flat (time)	1	0
Flat (external spatial)	$d-1$	$a,b,c,d,e,f,g,h,\ldots$

TABLE 4.1: Conventions for the indices used in this section.

Taking x^μ to be the coordinates of a d-dimensional manifold $\mathcal{M}_d$ we can take the first coordinate x^t to be the curved time coordinate and the remaining x^m to describe the spatial hypersurfaces.

Furthermore we want to write the theory in the *vielbein* formalism (which is sometimes also called the Cartan-, local frame- or tetrad formalism) where instead of using the metric as the dynamical field we use the frame field (or vielbein) $E_\mu{}^\alpha$. The vielbein formalism has some advantages over the metric formalism, namely that it makes the Lorentz symmetry manifest — which allows us to include this symmetry in the canonical analysis and that it is required if we ever want to extend the results and add Fermionic fields to the theory.

Using the Minkowski metric $\eta_{\alpha\beta}$ (with signature $-++\cdots+$) we can relate the vielbein to the metric via equation (4.19).

$$G_{\mu\nu} = E_\mu{}^\alpha E_\nu{}^\beta \eta_{\alpha\beta} \tag{4.19}$$

The inverse vielbein can be defined by the following equivalent identities.

$$E_\mu{}^\alpha E_\alpha{}^\nu = \delta_\mu^\nu \tag{4.20}$$
$$E_\beta{}^\mu E_\mu{}^\alpha = \delta_\beta^\alpha \tag{4.21}$$

Now we want to carry out the ADM decomposition of the space-time manifold and hence we have to introduce new degrees of freedom that are better adapted to this foliation of the manifold. On the spatial hypersurfaces we use a spatial vielbein $e_m{}^a$ or equivalently a spatial metric $g_{mn} := e_m{}^a e_n{}^b \delta_{ab}$. How different hypersurfaces in the foliation relate to each other is parametrised by the lapse function N and by the shift vector N^a — which will appear as Lagrange parameters of the time evolution and spatial diffeomorphisms in the Hamiltonian respectively. We define these new fields via the parametrisation of the vielbein as in equation (4.22).[5]

$$E_\mu{}^\alpha =: \begin{pmatrix} N & N^a \\ 0 & e_m{}^a \end{pmatrix} \tag{4.22}$$

Using the definition of the inverse vielbein (4.20) we find that it is parametrised by (4.23), where we defined $N^m := N^a\, e_a{}^m$. We raise and lower curved spatial indices with the spatial metric g_{mn}. Because we raise and lower flat spatial indices with the identity δ_{ab} the placement of these indices is irrelevant.

$$E_\alpha{}^\mu = \begin{pmatrix} N^{-1} & -N^{-1}N^m \\ 0 & e_a{}^m \end{pmatrix} \tag{4.23}$$

Using the relation (4.19) we can determine the parametrisation of the d-dimensional metric in terms of the new objects.

$$G_{tt} = N^a N_a - N^2 \tag{4.24}$$
$$G_{tn} = N_n = N_a e_n{}^a \tag{4.25}$$
$$G_{mn} = g_{mn} = e_m{}^a e_n{}^b \delta_{ab} \tag{4.26}$$

[5] A part of the Lorentz symmetry has to be gauge fixed in order for this parametrisation to be of triangular form.

Similarly we can find the parametrisation of the inverse metric using $G_{\mu\nu}G^{\nu\rho} = \delta^\rho_\mu$. Looking for example at equation (4.29) we can see that the parametrisation of the inverse metric is not componentwise the inverse of the metric.

$$G^{tt} = -N^{-2} \tag{4.27}$$
$$G^{tn} = N^{-2}N^n \tag{4.28}$$
$$G^{mn} = g^{mn} - N^{-2}N^m N^n \tag{4.29}$$

The determinant E of the d-dimensional vielbein decomposes according to (4.30) into the product of the lapse function and the spatial vielbein determinant e.

$$E = N \cdot e \tag{4.30}$$

The determinant G of the d-dimensional metric is given by (4.31). This is moreover important because it means we can express the vielbein determinant in terms of the metric as $E = \sqrt{-G}$.

$$G = -E^2 = -N^2 e^2 \tag{4.31}$$

When we are considering an action where further tensor fields, such as differential forms, exist on the manifold we also have to decompose them according to their space-time index structure. For example a one form A_μ becomes the fields A_t and A_m, a two form $B_{\mu\nu}$ becomes B_{tn} and B_{mn} etc. Scalar fields however remain unchanged.

We continue the discussion of the ADM formulation of general relativity in chapter 5, where we discuss the canonical formulation and analysis of the ungauged maximal $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity.

4.3 Canonical SL(n)/SO(n) scalar coset sigma model

In this section we look at the canonical formulation and analysis of scalar coset sigma models, taking the coset SL(n)/SO(n) as an example. The coset SL(n)/SO(n) is a relatively simple model described only by a single coset constraint — this means there is only one additional condition on a generic symmetric matrix of scalar fields M_{MN} to make it a coset representative. Nonetheless SL(n)/SO(n) will allow us to see the general structures we would find in more complicated coset models (e.g. $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$). We will go through the full canonical formalism of this model and in particular we will treat the coset constraint that is associated to SL(n)/SO(n) completely explicitly. In the end we will contrast this with the implicit treatment of the coset constraints.

4.3.1 Explicit treatment of the coset constraints

In order to describe the SL(n)/SO(n) scalar coset model we take indices $M, N = 1, ..., n$ and for simplicity only a time derivative in the scalar sigma model. We take an a priori general symmetric matrix of scalar fields $M_{MN} = M_{(MN)}$ but add to its kinetic term in the Lagrangian a coset constraint term in order to make M_{MN} a SL(n)/SO(n) representative, as in equation (4.32). For this model we only need to add the constraint $c := \det(M) - 1$ with a Lagrange multiplier ϕ. The inverse field is defined by $M_{MN}\, M^{NK} = \delta^K_M$.

$$\mathcal{L}(M_{MN}, \phi) = -\frac{1}{2}\dot{M}_{MN}\dot{M}^{MN} + c\,\phi \tag{4.32}$$

Since we only consider the time derivative we can immediately compute the canonical momenta of the scalar field and of the Lagrange multiplier field.

$$\Pi^{MN}(M) = -2\dot{M}^{MN} \tag{4.33}$$
$$\Pi(\phi) = 0 \tag{4.34}$$

By design we find the primary constraint (4.34), which will lead us to find the secondary constraint c. Defining $\Pi_{MN}(M) := -\Pi^{RS}(M)\, M_{RM}\, M_{SN}$ the canonical Hamiltonian is given by equation (4.35).

$$\mathcal{H} = +\frac{1}{8}\Pi_{MN}(M)\,\Pi^{MN}(M) + \dot{\phi}\,\Pi(\phi) - c\,\phi \tag{4.35}$$

To check the consistency of the primary constraint we need to construct the total Hamiltonian $\mathcal{H}_T$ as defined by adding the primary constraint with a general parameter to the canonical Hamiltonian $\mathcal{H}_T := \mathcal{H} + u\,\Pi(\phi)$.

$$\{\Pi(\phi), \mathcal{H}_T\} \overset{!}{=} 0 \tag{4.36}$$

Checking the consistency of the primary constraint with the condition (4.36) we find the secondary constraint (4.37) as expected.

$$c = \det(M) - 1 = 0 \tag{4.37}$$

We now need to repeat this process to check that the total time evolution of the secondary constraint vanishes, as is needed in order for the constraints to be consistent.

$$\{c, \mathcal{H}_T\} \overset{!}{=} 0 \tag{4.38}$$

In order to evaluate the condition (4.38) we need to use the fundamental Poisson bracket given by equation (4.39). This is the fundamental Poisson bracket for a generic symmetric Matrix and there is no coset projector term — this is due to the fact that M_{MN} is a coset representative only by virtue of the secondary constraint c, which we are not allowed to use before evaluating all Poisson brackets and hence this relation is the same for all cosets treated in this way. Furthermore we need to use the equation (4.40) which follows from the fundamental Poisson bracket identity.

$$\{M_{MN}(x), \Pi^{PQ}(M)(y)\} = \left(\delta^P_M\,\delta^Q_N + \delta^P_N\,\delta^Q_M\right)\delta(x-y) \tag{4.39}$$
$$\{\det(M), \Pi^{MN}(M)\} = \det(M)\, M^{MN} \tag{4.40}$$

We now find that the consistency condition (4.38) does indeed yield a tertiary constraint p given by equation (4.41).

$$\det(M)\, M_{MN}\,\Pi^{MN}(M) =: p = 0 \tag{4.41}$$

The tertiary constraint p, implying the tracelessness of the sl(n) algebra element $M_{RN}\,\Pi^{NS}(M)$, can be interpreted as a coset constraint on the momenta induced by the consistency of the coset constraint of the fields $\det(M) = 1$. The consistency of the tertiary constraint does not yield any new constraints and the consistency procedure thus terminates.

4.3.2 Second class constraints and the Dirac bracket

We can now compute the gauge algebra of the canonical constraints under the Poisson bracket. We find that the primary constraint Poisson-commutes with all other constraints, thus making it a first class constraint. The secondary and tertiary constraints however do not commute and form a second class system of constraints as given by equation (4.44). It is important to note here that the right hand side of equation (4.44) cannot be rewritten in terms of the secondary constraint without a constant term — as otherwise they would be first class constraints. The factor n originates from the trace of an SL(n) identity $M_{MN}\, M^{MN} = \delta^M_M = n$.

$$\{c, c\} = 0 \tag{4.42}$$

$$\{p, p\} = 0 \tag{4.43}$$

$$\{c, p\} = n \, \det(M)^2 \tag{4.44}$$

The fact that we have a system of second class constraints is equivalent to the fact that the algebra of constraints does not close under the Poisson bracket. The way to resolve this problem is to introduce the Dirac bracket $\{g, f\}_{DB}$ — where g and f are arbitrary functions on phase space — as defined in equation (4.45) which improves upon the Poisson bracket. Fortunately the equation (4.44) is a scalar equation and so it is straightforward to define the Dirac bracket.

$$\{f, g\}_{DB} := \{f, g\} - \frac{1}{n\, \det(M)^2} \left(\{f, p\}\{c, g\} - \{f, c\}\{p, g\}\right) \tag{4.45}$$

We find that all the constraints commute in the Dirac bracket and the algebra thus indeed closes.

We already mentioned that we are not allowed to apply the canonical constraints before evaluating all Poisson brackets and that hence the general result of equation (4.40) is in fact not inconsistent with the secondary constraint $\det(M) = 1$. The Dirac bracket does not have this restriction and we are free to apply the constraints before or after evaluating the Dirac bracket. Calculating the relation (4.40) with the Dirac bracket instead we find equation (4.46), which is clearly consistent with the constraint $\det(M) = 1$.

$$\{\det(M), \Pi^{MN}(M)\}_{DB} = 0 \tag{4.46}$$

4.3.3 Physical degrees of freedom and the implicit formalism

Now that we have a complete understanding of the algebra we can compute the number of physical degrees of freedom of this theory. The complete set of phase space variables is $\{M_{MN},\, \Pi^{KL}(M),\, \phi,\, \Pi(\phi)\}$. We need to count the components of M_{MN} and $\Pi^{KL}(M)$ as those of generic symmetric fields as the coset constraints are counted separately. We count $\frac{n(n+1)}{2} + \frac{n(n+1)}{2} + 1 + 1$ components for the phase space variables. There are three constraints, however the primary constraint is a first class constraint and thus has to be counted twice since it is associated to a gauge symmetry (the shift symmetry of the Lagrange multiplier) and we have effectively $2 \cdot 1 + 1 + 1$ constraints. Subtracting the constraints from the phase space field components leaves us with $n^2 + n - 2$ independent phase space degrees of freedom or equivalently with

$\frac{n^2}{2} + \frac{n}{2} - 1$ independent physical degrees of freedom in field space. This concludes the canonical analysis of the $\mathrm{SL}(n)/\mathrm{SO}(n)$ coset model.

In this canonical analysis we have treated the coset constraint $\det(M) = 1$ completely explicitly and seen that its introduction led to further coset constraints on the canonical momenta. Furthermore we saw that this explicit treatment of the coset constraints required us to introduce Dirac brackets. We call this treatment the *explicit* treatment of the coset constraints. However it may be quite difficult to write down all the coset constraints and go through these calculations for more complicated cosets, such as $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ and there is an implicit way of treating the coset constraints canonically.

The *implicit* treatment of coset constraints consists of using the scalar sigma model Lagrangian of a symmetric field, without adding the coset constraints explicitly, but applying the coset constraints only after evaluating all Poisson brackets. In a sense this implicit treatment resembles more closely how the constraints are normally treated in the Lagrangian formalism. In our simplified model the Lagrangian would hence be just the free theory $\mathcal{L} = -\frac{1}{2}\dot{M}_{MN}\dot{M}^{MN}$ and there would be no canonical constraints. The implicit formalism is only applicable if the coset constraints are not needed during the canonical analysis, since we are not allowed to apply them inside the Poisson brackets. The benefit of this formalism is that we can skip the entire analysis of the coset constraints — which may be of considerable effort for complicated cosets and may not provide much insight. Moreover we can still compute the right number of physical degrees of freedom by looking at the dimension of the coset — in the case of our model here it would be $\dim(\mathrm{SL}(n)/\mathrm{SO}(n)) = \frac{n^2}{2} + \frac{n}{2} - 1$, which confirms the result from the explicit analysis.

An alternative canonical treatment of coset sigma models, using the vielbein formalism, is described in reference [206].

4.4 Canonical topological 2-forms in $5 + 27$ dimensions

In this section we investigate the canonical formulation of a model theory of certain topological generalised two-forms $B_{\mu\nu M}$ on the $(5 + 27)$-dimensional extended generalised exceptional geometry (cf. section 3.5.2). We want to consider this model theory, based on the topological kinetic term of the two-forms of the $\mathrm{E}_{6(6)}$ ExFT, as a preparation to the canonical analysis of the full ExFT. Due to its topological nature and the use of the extended generalised geometry, this model theory has a somewhat unusual canonical structure, which we seek to explore and better understand before adding further complications.

Starting from the Lagrangian of the model — which we introduce below — we first calculate the canonical momenta of the two-forms and the canonical Hamiltonian in section 4.4.1. In section 4.4.2 we construct the complete set of canonical constraints. Next we compute all canonical transformations and the full algebra of the constraints in section 4.4.3. We thereby confirm that no propagating degrees of freedom exist in this theory. Moreover we learn that the external diffeomorphisms are not canonically generated in this model, because of the topological nature of the two-forms. In section 4.4.4 we identify some obstacles, which appear for constraint algebras of a certain form in exceptional generalised geometry, related to the construction of Dirac brackets.

In this section we use the notation introduced for the Lagrangian formulation of the $E_{6(6)}$ ExFT in section 3.5. We want to consider an action of the form (4.47) and we denote the five external coordinates by x^μ and the 27 internal coordinates by Y^M. As was discussed in section 3.5, the integral over the internal coordinates of the exceptional generalised geometry in the action (4.47) is taken to be symbolic, because we do not know how this integral should be carried out explicitly while the section condition is being observed.

$$S_B = \int d^5x\, d^{27}Y\, \mathcal{L}_B \tag{4.47}$$

The Langrangian of the model we want to examine is given by (4.48), where ϱ is an overall real constant and $\epsilon^{\mu\nu\rho\sigma\tau}$ is the constant Levi-Civita symbol in five dimensions.

$$\mathcal{L}_{\mathrm{B}} = -\varrho\, \epsilon^{\mu\nu\rho\sigma\tau}\, d^{MNR}\, \partial_R B_{\mu\nu M}\, \partial_\rho B_{\sigma\tau N} \tag{4.48}$$

In the two-form kinetic term of the full ExFT the external derivative in (4.48) is covariantised, however the issues that we want to discuss in this section are unrelated to this covariantisation. Because we ignore this covariantisation term we do not see the generalised diffeomorphisms as canonical transformations in this model theory.

$$H_{\rho\sigma\tau N} := 3\, \partial_{[\rho} B_{\sigma\tau]N} \tag{4.49}$$

There are several alternative ways of rewriting the Lagrangian and action of this model. We can make use of the (naive) two-form field strength $H_M := \mathrm{d}B_M$ (4.49) to rewrite (4.48) as either (4.50) or equivalently (4.51). We should note that this Lagrangian is linear in the field strength H_M and hence linear in the time derivative. This linearity in the time derivative, combined with the $d^{MNK}\,\partial_K$ projector that appears in the Lagrangian, leads to the peculiar canonical structure that we will describe in the following analysis.

$$\begin{aligned}\mathcal{L}_{\mathrm{B}} &= -\frac{\varrho}{30}\, d^{MNR}\, \partial_R B_M \wedge H_N &\quad (4.50)\\ &= -\frac{\varrho}{3}\, \epsilon^{\mu\nu\rho\sigma\tau}\, d^{MNR}\, \partial_R B_{\mu\nu M}\, H_{\rho\sigma\tau N} &\quad (4.51)\end{aligned}$$

Moreover we can reformulate the action (4.47) equivalently as the boundary term (4.52) in a $(6+27)$-dimensional geometry. Because the two-form field strength is closed, i.e. $\mathrm{d}H_M = 0$, the rewriting (4.52) is equal to (4.53), which can be compared to the two-form dependent term in the ExFT topological term (3.105).

$$\begin{aligned}S_B &= -\frac{\varrho}{30}\int d^6\tilde{x}\, d^{27}Y\, \mathrm{d}\left(d^{MNR}\, \partial_R B_M \wedge H_N\right) &\quad (4.52)\\ &= -\frac{\varrho}{30}\int d^6\tilde{x}\, d^{27}Y\, d^{MNR}\, \partial_R H_M \wedge H_N &\quad (4.53)\end{aligned}$$

The external five-dimensional indices decompose in a $(1+4)$-dimensional space-time split as $\mu = (t, m)$, where t denotes the (curved) time index and $m = 1, \ldots, 4$ denotes the (curved) spatial index. Decomposing the space-time indices of the two-form and the Levi-Civita symbol we can write the Lagrangian as (4.55), which is the starting point of the canonical formalism.

$$\begin{aligned}
\mathcal{L}_{\mathrm{B}} &= -\,\varrho\,\epsilon^{\mu\nu\rho\sigma\tau}\,d^{MNR}\,\partial_R B_{\mu\nu M}\,\partial_\rho B_{\sigma\tau N} && (4.54)\\
&= -\,\varrho\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\partial_t B_{slN} && (4.55)\\
&\quad -2\varrho\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{tnM}\,\partial_r B_{slN}\\
&\quad +2\varrho\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\partial_s B_{tlN}
\end{aligned}$$

4.4.1 Canonical momenta and canonical Hamiltonian

The canonical momenta of the time component $\Pi^{tlN}(B)$ and the spatial components $\Pi^{slN}(B)$ of the two-forms can be read off from the space-time split of the Lagrangian (4.55) and are given explicitly by (4.56) and (4.57). Because the Lagrangian is linear in the time derivative ∂_t the canonical momenta do not contain any time derivatives themselves, which means that each of the canonical momenta leads to a primary constraint. We denote the primary constraints by $\mathcal{H}_{\mathrm{P1}}$ and $\mathcal{H}_{\mathrm{P2}}$ and they are defined by (4.56) and (4.57).

$$\begin{aligned}
(\mathcal{H}_{\mathrm{P1}})^{lN} &:= \Pi^{tlN}(B) = 0 && (4.56)\\
(\mathcal{H}_{\mathrm{P2}})^{slN} &:= \Pi^{slN}(B) + 2\varrho\,\epsilon^{tmnsl}\,d^{MNR}\,\partial_R B_{mnM} = 0 && (4.57)
\end{aligned}$$

The Legendre transformation of the Lagrangian (4.55) is given by (4.58). Because we are summing over all of the spatial components of an antisymmetric object we need to insert a factor of 1/2 in the transformation in order to avoid overcounting.

$$\begin{aligned}
\mathcal{H}_{\mathrm{B}} &= \dot{B}_{tlN}\cdot\Pi^{tlN}(B) + \frac{1}{2}\,\dot{B}_{mnN}\cdot\Pi^{mnN}(B) - \mathcal{L}_{\mathrm{B}} && (4.58)\\
&= \dot{B}_{tlN}\cdot(\mathcal{H}_{\mathrm{P1}})^{lN} + B_{tnM}\cdot(\mathcal{H}_{\mathrm{S1}})^{nM} && (4.59)
\end{aligned}$$

Using the space-time split of the Lagrangian, the definition of the canonical momenta and the primary constraints we can write the canonical Hamiltonian as (4.59). Here we have furthermore introduced $\mathcal{H}_{\mathrm{S1}}$, which are explicitly defined by (4.60) and will turn out to be secondary constraints.

$$(\mathcal{H}_{\mathrm{S1}})^{nM} := -4\varrho\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R\partial_r B_{slN} \qquad (4.60)$$

We can use the field strength (4.49) to rewrite (4.60) as (4.61).

$$(\mathcal{H}_{\mathrm{S1}})^{nM} = -\frac{4\varrho}{3}\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R H_{rslN} \qquad (4.61)$$

4.4.2 Constraints and consistency

The consistency of the primary constraints (4.56) and (4.57) has to be verified. The consistency of constraints is equivalent to a vanishing time evolution with respect to the total Hamiltonian $\mathcal{H}_{\text{B-Total}}$ (4.62). The total Hamiltonian is defined as the canonical Hamiltonian plus a general linear phase space sum over the primary constraints.

$$\mathcal{H}_{\text{B-Total}} := \mathcal{H}_{\mathrm{B}} + (u_1)_{lN}\cdot(\mathcal{H}_{\mathrm{P1}})^{lN} + (u_2)_{slN}\cdot(\mathcal{H}_{\mathrm{P2}})^{slN} \qquad (4.62)$$

Next we need to define the fundamental equal time Poisson brackets. Using the notation that $X_1 := (x_1, Y_1)$ denotes both the spatial external and internal coordinates and writing $X_1 - X_2 = (x_1 - x_2, Y_1 - Y_2)$ we can define the fundamental equal time Poisson brackets by (4.63) and (4.64), where $\delta^{(4+27)}(X_1 - X_2)$ is the $(4+27)$-dimensional Dirac delta distribution.

$$\{B_{tlR}(X_1), \Pi^{tnS}(B)(X_2)\} = \delta_l^n \delta_R^S \, \delta^{(4+27)}(X_1 - X_2) \tag{4.63}$$
$$\{B_{klR}(X_1), \Pi^{mnS}(B)(X_2)\} = (\delta_k^m \delta_l^n - \delta_l^m \delta_k^n) \, \delta_R^S \, \delta^{(4+27)}(X_1 - X_2) \tag{4.64}$$

It is computationally advantageous to first compute the algebra of the primary constraints before considering their consistency. We find that all primary constraints Poisson-commute.

$$\{(\mathcal{H}_{\mathrm{P1}})^{kK}, (\mathcal{H}_{\mathrm{P1}})^{lL}\} = 0 \tag{4.65}$$
$$\{(\mathcal{H}_{\mathrm{P1}})^{kK}, (\mathcal{H}_{\mathrm{P2}})^{mnM}\} = 0 \tag{4.66}$$
$$\{(\mathcal{H}_{\mathrm{P2}})^{klK}, (\mathcal{H}_{\mathrm{P2}})^{mnM}\} = 0 \tag{4.67}$$

This means that their time time evolution generated by the total Hamiltonian is equivalent to the time evolution generated by the canonical Hamiltonian (4.59).

The consistency condition of the primary constraints $\mathcal{H}_{\mathrm{P1}}$ is given by (4.68) and confirms that the $(\mathcal{H}_{\mathrm{S1}})$ (4.61) are indeed secondary constraints.

$$\{(\mathcal{H}_{\mathrm{P1}})^{kK}, \mathcal{H}_{\text{B-Total}}\} = -(\mathcal{H}_{\mathrm{S1}})^{nM} = 0 \tag{4.68}$$

It is convenient to define the smeared or integrated constraints $\mathcal{H}_{\mathrm{S1}}[\Phi]$ by (4.69), with Φ_{nM} being the (test function) smearing parameters. This allows us to move derivatives onto the parameters instead of writing derivatives of Dirac delta distributions.

$$\mathcal{H}_{\mathrm{S1}}[\Phi] := \int d^4x \, d^{27}Y \; \Phi_{nM} \cdot (\mathcal{H}_{\mathrm{S1}})^{nM} \tag{4.69}$$

It is furthermore useful to first calculate the Poisson brackets of $\mathcal{H}_{\mathrm{S1}}[\Phi]$ with the primary constraints, before we go on to check the consistency of the remaining primary constraints $\mathcal{H}_{\mathrm{P2}}$. We find that $\mathcal{H}_{\mathrm{P1}}$ Poisson-commutes with the $\mathcal{H}_{\mathrm{S1}}[\Phi]$, whereas $\mathcal{H}_{\mathrm{P2}}$ does not. Therefore $\mathcal{H}_{\mathrm{P2}}$ and $\mathcal{H}_{\mathrm{S1}}$ are second class constraints.

$$\{(\mathcal{H}_{\mathrm{P1}})^{mM}, \mathcal{H}_{\mathrm{S1}}[\Phi]\} = 0 \tag{4.70}$$
$$\{(\mathcal{H}_{\mathrm{P2}})^{mnM}, \mathcal{H}_{\mathrm{S1}}[\Phi]\} = 8\varrho \, \epsilon^{tklmn} \, d^{KLM} \, \partial_K \partial_l \Phi_{kL} \tag{4.71}$$

Using the explicit expression for the Poisson bracket (4.71) it is straightforward to compute the consistency condition (4.72) of the primary constraints $\mathcal{H}_{\mathrm{P2}}$. The condition (4.72) implies the existence of the secondary constraints $\mathcal{H}_{\mathrm{S2}}$, which are of the same form as expression (4.71) but with B_{tkL} replacing the smearing parameters.

$$\{(\mathcal{H}_{\mathrm{P2}})^{mnM}, \mathcal{H}_{\text{B-Total}}\} =: (\mathcal{H}_{\mathrm{S2}})^{mnM} = 8\varrho \, \epsilon^{tklmn} \, d^{KLM} \, \partial_K \partial_l B_{tkL} = 0 \tag{4.72}$$

Finally we have to verify the consistency of the secondary constraints, but no new constraints arise and the consistency algorithm thus terminates. The coefficient functions of the total Hamiltonian however should be vanishing $u_1 \equiv 0$, $u_2 \equiv 0$. The complete set of canonical constraints of the model is listed below.

$$(\mathcal{H}_{\mathrm{P1}})^{lN} = \Pi^{tlN}(B) \tag{4.73}$$

$$(\mathcal{H}_{\mathrm{P2}})^{slN} = \Pi^{slN}(B) + 2\varrho\, \epsilon^{tmnsl}\, d^{MNR}\, \partial_R B_{mnM} \tag{4.74}$$

$$(\mathcal{H}_{\mathrm{S1}})^{nM} = -4\varrho\, \epsilon^{tnrsl}\, d^{KLM}\, \partial_K \partial_r B_{slL} \tag{4.75}$$

$$(\mathcal{H}_{\mathrm{S2}})^{mnM} = +8\varrho\, \epsilon^{tklmn}\, d^{KLM}\, \partial_K \partial_l B_{tkL} \tag{4.76}$$

4.4.3 Canonical transformations, algebra and degrees of freedom

In the following we make use of the integrated versions of all constraints. In the context of canonical transformations the smearing parameters are interpreted as the parameters of the canonical (gauge) transformations. All non-vanishing transformations generated by the canonical constraints are listed below.

$$\{B_{tnN}, \mathcal{H}_{\mathrm{P1}}[\chi_1]\} = 2\,(\chi_1)_{nN} \tag{4.77}$$

$$\{B_{mnN}, \mathcal{H}_{\mathrm{P2}}[\chi_2]\} = 2\,(\chi_2)_{mnN} \tag{4.78}$$

$$\{\Pi^{mnN}(B), \mathcal{H}_{\mathrm{P2}}[\chi_2]\} = +4\varrho\, \epsilon^{tmnsl}\, d^{MNR}\, \partial_R (\chi_2)_{slM} \tag{4.79}$$

$$\{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{S1}}[\Phi_1]\} = +8\varrho\, \epsilon^{tlkmn}\, d^{KLM}\, \partial_K \partial_k (\Phi_1)_{lL} \tag{4.80}$$

$$\{\Pi^{tnM}(B), \mathcal{H}_{\mathrm{S2}}[\Phi_2]\} = +16\varrho\, \epsilon^{tnklm}\, d^{KLM}\, \partial_K \partial_k (\Phi_2)_{lmL} \tag{4.81}$$

Because the field strengths of differential form fields are defined by the exterior derivative there are no time derivatives on the time components of the differential forms. And if the Lagrangian is furthermore linear in the field strength, then all the primary canonical constraints directly relate the field components to the canonical momenta. This then leads to the generic shift transformations (4.77) and (4.78) generated by the primary constraints. These general shift transformations in particular include transformations where the gauge parameter is a derivative, e.g. $(\chi_2)_{mnN} =: \partial_{[m}\lambda_{n]N}$, which leads to the more familiar transformations $\delta_{\mathcal{H}_{\mathrm{P2}}[\lambda]} B_{mnN}$ given by (4.82).

$$\delta_{\mathcal{H}_{\mathrm{P2}}[\lambda]} B_{mnN} = \{B_{mnN}, \mathcal{H}_{\mathrm{P2}}[\lambda]\} = 2\,\partial_{[m}\lambda_{n]N} \tag{4.82}$$

In the case of free Maxwell theory the general shift transformations of the one-form can be made into the usual form by way of the parameters of the extended Hamiltonian (see chapter 19 of reference [28]). As we will determine below, all of the constraints in our model are second class and therefore the extended Hamiltonian agrees with the already fully determined total Hamiltonian. It is hence unclear how a procedure analogous to that of reference [28] would work in this model. It seems nonetheless probable that a way of fully rearranging the shift transformations into the usual form should exist (and that these transformations survive the introduction of the Dirac bracket). Moreover it may be instructive to investigate the canonical formulation of the three-dimensional Chern-Simons theory $\mathcal{L} = A \wedge F$, whose canonical constraints are structured in a similar way as those of our model theory, because it is also topological and linear in the field strength.

The action of this model can be written in terms of differential forms as the boundary integral (4.52). We should therefore expect that the (external) diffeomorphisms are a symmetry of this action — but canonically we do not find any constraints that generate diffeomorphism transformations. The (external) diffeomorphisms are usually generated by the secondary constraints that arise from the consistency requirement of the primary constraints $\Pi^a(N_a) = 0$, which tell us that the canonical momenta of the shift vector are vanishing. In a purely topological theory the fields do not couple to the space-time metric by definition and therefore we do not see any external diffeomorphism transformations in purely topological theories, including in this model. Even if we couple this model to other fields, as in ExFT, we do not see any external diffeomorphism transformation as long as the only kinetic term is topological. By contrast the internal generalised diffeomorphism symmetry appears canonically if we covariantise the external derivative of the model with respect to the generalised diffeomorphisms, as is discussed in chapter 6. The (non-)existence of Lagrangian symmetries in the canonical formalism has also been discussed in [207].

Now that we have all the non-vanishing canonical transformations we can compute the full algebra of the canonical constraints. The primary and the secondary constraints Poisson-commute amongst each other and there are only two non-vanishing Poisson brackets, which mix primary and secondary constraints. The relation (4.83) is what we have already seen in equation (4.71), the other relation can be stated as (4.84).

$$\{\mathcal{H}_{\mathrm{P1}}[\chi_1], \mathcal{H}_{\mathrm{S2}}[\Phi_2]\} = +16\varrho\, \epsilon^{tklmn}\, d^{KLM}\, (\chi_1)_{kL}\, \partial_K \partial_l (\Phi_2)_{mnM} \tag{4.83}$$

$$\{\mathcal{H}_{\mathrm{P2}}[\chi_2], \mathcal{H}_{\mathrm{S1}}[\Phi_1]\} = +8\varrho\, \epsilon^{tklmn}\, d^{KLM}\, (\chi_2)_{mnM}\, \partial_K \partial_l (\Phi_1)_{kL} \tag{4.84}$$

All canonical constraints are involved in these two relations, which do not close into the canonical constraints. Therefore all of the canonical constraints of this model are second class functions.

#	Fields	Momenta	Primary	Secondary
108	B_{tnN}	Π^{tnN}	$(\mathcal{H}_{\mathrm{P1}})^{nN}$	$(\mathcal{H}_{\mathrm{S1}})^{nN}$
162	B_{mnN}	Π^{mnN}	$(\mathcal{H}_{\mathrm{P2}})^{mnN}$	$(\mathcal{H}_{\mathrm{S2}})^{mnN}$

TABLE 4.2: This table lists the number and names of the independent components of all the fields, canonical momenta as well as of the primary and secondary constraints.

As is expected in a topological theory, there are no propagating degrees of freedom and the number of physical degrees of freedom described by the Lagrangian (4.48) is zero. In the canonical formalism this is because the total number of independent components of all the second class constraints $\mathcal{H}_{\mathrm{P1}}$, $\mathcal{H}_{\mathrm{P2}}$, $\mathcal{H}_{\mathrm{S1}}$ and $\mathcal{H}_{\mathrm{S2}}$ exactly cancels the number of independent phase space variables of the theory, as can be seen in table 4.2.

4.4.4 Dirac brackets in exceptional generalised geometry

Due to the existence of second class constraints the canonical analysis should proceed with the construction of the Dirac bracket $\{.,.\}_{\mathrm{DB}}$. Taking $a, b \in \{P1, P2, S1, S2\}$ to be symbolic indices that label the constraints we can define the matrix M_{ab} by (4.85).

$$M_{ab}(x_1, x_2, Y_1, Y_2) := \{\mathcal{H}_a(x_1, Y_1), \mathcal{H}_b(x_2, Y_2)\} \tag{4.85}$$

The components of the matrix M_{ab} are identical to the constraint algebra relations above, but with the smearing parameters replaced by Dirac delta distributions. Indices that were contracted into the smearing parameters are now open, but we take them to be included in the symbolic indices a, b too.

Using (4.85) we can now try to define the Dirac bracket of this model by (4.86), however there are several difficulties and potential problems with this definition.

$$\{f,g\}_{\mathrm{DB}} := \{f,g\} - \sum_{a,b} \int d^4x_1 \int d^4x_2 \int d^{27}Y_1 \int d^{27}Y_2 \\ \cdot \Big(\{f, \mathcal{H}_a(x_1,Y_1)\}\, M^{ab}(x_1,x_2,Y_1,Y_2)\, \{\mathcal{H}_b(x_2,Y_2), g\} \Big) \qquad (4.86)$$

The first difficulty concerns the definition of the inverse matrix M^{ab}. Because the components of M_{ab} depend on Dirac delta distributions and have open indices its inverse should be defined by a condition such as (4.87).

$$\sum_b \int d^4x_2 \int d^{27}Y_2\, M_{ab}(x_1,x_2,Y_1,Y_2)\, M^{bc}(x_2,x_3,Y_2,Y_3) \\ = \delta_a{}^c\, \delta^{(4+27)}(x_1 - x_3, Y_1 - Y_3) \qquad (4.87)$$

Because of the derivatives in the constraint algebra relations (4.83) and (4.84), solving equation (4.87) for the components of the inverse M^{ab} requires us to find distributions Ψ, which satisfy equations of the type (4.88), where mixed external and internal derivatives of Ψ yield the $(4+27)$-dimensional Dirac delta distribution.

$$(\dots)^{mM}\, \partial_M \partial_m \Psi(x_1 - x_3, Y_1 - Y_3) = \delta^{(4+27)}(x_1 - x_3, Y_1 - Y_3) \qquad (4.88)$$

Thus solving equations of the form of (4.87), to determine the inverse M^{ab}, turns out to be a rather difficult problem, because we need to find a primitive function of the $(4+27)$-dimensional Dirac delta distribution. This problem arises in all constraint algebras of a form that includes terms with mixed external and internal derivatives. Because the $\mathrm{E}_{6(6)}$ d-symbol and the Levi-Civita symbol are invertible one should be able to solve (4.87) if such a distribution can be identified.

Moreover it is not obvious that this definition of the Dirac bracket is well defined and that the internal integration can be carried out consistently, because the internal integrals in (4.86) and (4.87) have to be carried out while observing the section condition.

In principle it should be possible for us to avoid the introduction of a Dirac bracket altogether by "unfixing" the gauge conditions that make the constraints of this model second class functions [28, 208]. However the procedure of introducing a new set of first class constraints, together with additional gauge fixing conditions, to replace the second class constraints, is not unique and it is not immediately clear how one should proceed with this model.

We should again point out that the structure of the canonical constraints in this model is very similar to the canonical structure of the three-dimensional Chern-Simons theory $\mathcal{L} = A \wedge F$, which is also topological and linear in the field strength. It may thus be possible to identify a solution to these problems in Chern-Simons theory. There is of course no analogue to the generalised geometry in ordinary Chern-Simons theory, although it may be possible to add an internal derivative to Chern-Simons theory with a suitably symmetric symbol, which would make the theories even more similar.

In reference [209] Dirac brackets have recently been used in the context of the canonical formulation of exceptional world volume theories, with a definition somewhat similar to (4.86), but in a very different set up.

Chapter 5

Canonical $E_{6(6)}(\mathbb{R})$ invariant five-dimensional (super-)gravity

In this chapter we investigate the canonical formulation and analysis of the bosonic sector of the unique ungauged maximal five-dimensional supergravity theory that is manifestly invariant under the global action of $E_{6(6)}(\mathbb{R})$. This chapter is based on and closely follows the structure of parts of the publication [40].

As was discussed in section 2.3, the compactification of eleven-dimensional supergravity on an n-torus leads to a symmetry enhancement and the lower dimensional theory gains a hidden global $E_{6(6)}(\mathbb{R})$ symmetry. The Lagrangian formulation of the manifestly $E_{6(6)}(\mathbb{R})$ invariant five-dimensional supergravity theory was first described in [74]. Equivalently one can reach the manifestly $E_{6(6)}(\mathbb{R})$ invariant theory directly by choosing the trivial solution to the section condition of the $E_{6(6)}$ exceptional field theory [25], which was discussed in section 3.5.

The five-dimensional manifestly $E_{6(6)}(\mathbb{R})$ invariant supergravity theory is an interesting theory in its own right and we should expect there to be applications of the results of this analysis that are unrelated to exceptional field theory. For the purpose of this thesis however we are mainly interested in the five-dimensional theory as a stepping stone towards the canonical analysis of the full $E_{6(6)}$ exceptional field theory.
In the canonical analysis of the five-dimensional supergravity the main computational difficulties are the treatment of the Einstein-Hilbert term and the topological term of the one-forms. The study of the (relatively simple) topological term of this model gives crucial insights into the canonical treatment of topological terms of this form, which will be very useful in the analysis of the exceptional field theory. Conceptually one also has to pay attention to the treatment of the scalar coset model, which has already been discussed in section 4.3 for the example of the $SL(n)/SO(n)$ coset.
What makes this theory a good theory to consider in view of the $E_{6(6)}$ exceptional field theory, is that much of the structure of these theories is indeed analogous. What is missing in the five-dimensional theory is the generalised exceptional geometry, hence also the generalised exceptional diffeomorphisms and the topological two-forms, which were analysed separately in section 4.4.[1]

This chapter is structured as follows. In section 5.1 we review the Lagrangian formulation [74] of the five-dimensional ungauged maximal $E_{6(6)}(\mathbb{R})$ invariant supergravity theory. In section 5.2 we construct the canonical formulation of this theory and calculate the canonical momenta, the full set of the canonical constraints and the canonical

[1] Alternatively one could also consider the canonical formulation of the gauged maximal five-dimensional supergravity theory [84], which is in some ways even more similar to the $E_{6(6)}$ ExFT, but it also just as complicated as the ExFT and therefore not much of a simplification.

Hamiltonian. The canonical formulation is then analysed in section 5.3, where we compute all gauge transformations, the complete gauge algebra formed by the canonical constraints and rederive the number of physical degrees of freedom.

5.1 $\mathrm{E}_{6(6)}(\mathbb{R})$ invariant five-dimensional (super-)gravity

In this section we review the Lagrangian formulation [74] of the bosonic sector of the maximal ungauged supergravity in five dimensions, that is also manifestly $\mathrm{E}_{6(6)}(\mathbb{R})$ invariant, before we carry out its canonical formulation in the following sections.

We take the indices $M, N = 1, \ldots, 27$ to be in the (anti-)fundamental representation of $\mathrm{E}_{6(6)}(\mathbb{R})$. The curved five-dimensional indices $\mu, \nu = t, 1, \ldots, 5$ decompose in the space-time split as $\mu = (t, m)$ and the flat five-dimensional Lorentz indices $\alpha, \beta = 0, \ldots, 5$ decompose in the space-time split as $\alpha = (0, a)$.

The bosonic field content of the theory is $\{G_{\mu\nu}, A_\mu^M, M_{MN}\}$ — equivalently we can describe this on the ADM foliation of space-time as $\{N, N^a, e_m{}^a, A_t^M, A_m^M, M_{MN}\}$ [15, 74]. As described in section 4.2 the five-dimensional metric $G_{\mu\nu}$ decomposes into the lapse function N, the shift vector field N^a and the spatial vielbein $e_m{}^a$. The generalised one form fields A_μ^M are $\mathrm{U}(1)^{27}$ abelian vector gauge fields with one Lorentz index and one fundamental $\mathrm{E}_{6(6)}(\mathbb{R})$ index. The fields M_{MN} are Lorentz scalars and carry two symmetric fundamental $\mathrm{E}_{6(6)}(\mathbb{R})$ indices.

We want the scalars to describe the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset model and hence they need to be elements of this 42-dimensional coset [74]. The scalars transform covariantly under the global action of $\mathrm{E}_{6(6)}(\mathbb{R})$ and are invariant under the local action of $\mathrm{USp}(8)$. We can interpret the scalar fields as an $\mathrm{E}_{6(6)}(\mathbb{R})$ metric. If we wanted to add the fermionic fields to our analysis it would be necessary to describe the scalar fields in terms of the coset vielbein $\mathcal{V}_M^{AB}$ with $\mathrm{USp}(8)$ indices A, B [74].

The symmetric scalar matrix M_{MN} $(M, N = 1, ..., 27)$ has 378 components, but since it has to parametrise the 42-dimensional coset $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ not all components of M_{MN} are independent and there is a large number of coset constraints. We have seen in section 4.3 — for the much simpler coset $\mathrm{SL}(n)/\mathrm{SO}(n)$ — how one can add the coset constraints to the Lagrangian explicitly and then go through the analysis of their second class system of constraints. In the explicit treatment of the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset we would also need to write down explicitly all the coset constraints and add them to the Lagrangian. They would then produce many canonical constraints whose consistency conditions would produce further constraints on the canonical momenta of M_{MN}. These constraints would form systems of second class constraints and require the introduction of the Dirac bracket. Due to the much greater complexity of the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset we want to employ the implicit treatment of the coset constraints, which we also explained in section 4.3. We treat the scalar fields M_{MN} as an a priori generic symmetric matrix until all Poisson brackets have been fully evaluated. Only at that point we can apply the coset constraints — since they should still be thought of as canonical constraints we cannot apply them inside the Poisson brackets. The implicit formalism is sufficient for the calculations that we need to carry out and we find that it leads to the correct gauge transformations and dynamics.

Now that we know what the field content of the theory is we can write down the action of the theory as given by the equation (5.1). The Lagrangian density can be written as in equation (5.2). This Lagrangian was first published in 1980 [74]. The Lagrangian can also be found by taking the Lagrangian of the $\mathrm{E}_{6(6)}(\mathbb{R})$ ExFT and then applying the trivial solution of the section condition $\partial_M = 0$ [25].

$$S = \int d^5x \; \mathcal{L}_{5\mathrm{D}} \tag{5.1}$$

$$\begin{aligned} \mathcal{L}_{5\mathrm{D}} = &+ E\ R_5 + \frac{1}{24} E\ G^{\mu\nu}\ \partial_\mu M_{MN}\ \partial_\nu M^{MN} \\ &- \frac{1}{4} E\ M_{MN}\ F^M_{\mu\nu}\ F^N_{\rho\sigma}\ G^{\mu\rho}\ G^{\nu\sigma} + \kappa_5\ \epsilon^{\mu\nu\rho\sigma\tau}\ d_{LMN}\ A^L_\mu\ F^M_{\nu\rho}\ F^N_{\sigma\tau} \end{aligned} \tag{5.2}$$

The Lagrangian consists of the Einstein-Hilbert term in five dimensions, the scalar coset sigma model of $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$, a $\mathrm{U}(1)^{27}$ Maxwell theory term of the generalised one forms and a topological term. The fields are minimally coupled to the metric. $R_5 = (R_5)^\mu{}_{\nu\mu\sigma}\, G^{\nu\sigma}$ is the Ricci scalar in five dimensions. The inverse scalar fields M^{MN} are defined by requiring $M_{MP}M^{PN} = \delta^N_M$.

The A^M_μ kinetic term is a Maxwell theory type term where the additional indices of the $\mathrm{E}_{6(6)}(\mathbb{R})$ representation are contracted by the scalar fields. The abelian field strength is $F^M_{\mu\nu} := 2\ \partial_{[\mu} A^M_{\nu]}$.[2] Unlike in standard Maxwell theory the abelian gauge group of the one forms is $\mathrm{U}(1)^{27}$ since they carry the additional anti-fundamental representation $\overline{27}$ of $\mathrm{E}_{6(6)}(\mathbb{R})$.

The topological term $\sim A{\wedge}F{\wedge}F$ is metric independent and second order in derivatives. Its coefficient $\kappa_5 = +\frac{\sqrt{10}}{24}$ (the sign is convention dependent) is determined by the requirement of maximal supersymmetry in five dimensions.[3] Furthermore this value of κ_5 guarantees $\mathrm{E}_{8(8)}(\mathbb{R})$ invariance when reducing to three dimensions (cf. reference [210] for the reduction from eleven dimensions), however for the following analysis we do not need the precise value. We can relate the coefficient κ_5 also to the coefficient of the topological term of $\mathrm{E}_{6(6)}$ ExFT (3.135) by $\kappa_5 = \frac{1}{4}\kappa$ — the factor 1/4 originates from the use of the abelian field strength in (5.2).

5.2 Canonical formulation: Hamiltonian and canonical constraints

Following the description of the Lagrangian theory in section 5.1 we can now start to investigate the canonical formulation of the theory. We begin by finding the canonical momenta and then calculate the canonical Hamiltonian. This allows us to construct the total Hamiltonian and identify the complete and consistent set of canonical constraints.

[2] The Bianchi identity $\partial_{[\mu}\, F^M_{\nu\rho]} = 0$ is not a canonical constraint however since it does not constrain the canonical variables. The Bianchi identity follows directly from the commutativity of the partial derivatives.

[3] Without the requirement of maximal supersymmetry other values of κ_5 can be considered. In reference [210] the ungauged minimal supergravity theory with the value $\kappa_5 = \pm\frac{1}{3\sqrt{3}}$ is investigated This theory does not have a scalar field and only has a single gauge field. Upon compactification to three dimensions this theory yields a G_2 symmetry

5.2.1 Canonical momenta

Defining the coefficients of anholonomy $\Omega_{\alpha\beta\gamma} := 2\, E_{[\alpha}{}^{\mu}\, E_{\beta]}{}^{\nu}\, \partial_\mu E_{\nu\gamma}$ we can write the ADM decomposition of the Einstein-Hilbert term as in equation (5.3) [205].

$$E\, R_5 = N\, e\, \left(R_4 + \Omega_{0(ab)}\, \Omega_{0(ab)} - \Omega_{0aa}\Omega_{0bb}\right) \tag{5.3}$$

In particular the Ω_{0bc} components of the coefficients of anholonomy are the only components that contain a time derivative — notably this derivative only acts on the spatial vielbein. Explicitly we can write these components as (5.4).

$$\Omega_{0bc} = N^{-1} \cdot \left[e_b{}^n(\partial_t - N^m\partial_m)e_{nc} - e_b{}^m e_{nc}\partial_m N^n\right] \tag{5.4}$$

Because the Lagrangian does not depend on the time derivatives of the lapse function, the shift vector and the time component of the gauge field the following shift type primary constraints of the form $\Pi(X) = 0$ have to exist.

$$\Pi(N) = 0 \tag{5.5}$$
$$\Pi^a(N_a) = 0 \tag{5.6}$$
$$\Pi_M(A_t^M) = 0 \tag{5.7}$$

Moreover we find the following non-vanishing canonical momenta.

$$\Pi^m{}_a(e) = +\, 2\, e\, e^{bm}\left[\Omega_{0(ab)} - \delta_{ab}\Omega_{0cc}\right] \tag{5.8}$$
$$\Pi^l_T(A) = +\frac{e}{N} g^{ln}\, M_{TN}\, \left[F^N_{tn} + N^k F^N_{nk}\right] + 4\kappa_5\, \epsilon^{lmnr} d_{MNT}\, A^M_m\, F^N_{nr} \tag{5.9}$$
$$\Pi^{RS}(M) = +\frac{1}{6}\frac{e}{N}\, \left[\dot{M}_{QP}\, M^{QR} M^{PS} + N^n\, \partial_n M^{RS}\right] \tag{5.10}$$

The $\Pi_{ab}(e)$ and $\Pi(e)$ are contractions of the spatial vielbein momentum with the vielbein defined as follows.

$$\Pi_{ab}(e) := +\, e_{m(a}\, \Pi^m{}_{b)}(e) \tag{5.11}$$
$$\Pi(e) := +\, e_m{}^a\, \Pi^m{}_a(e) \tag{5.12}$$

In the general relativity literature (e.g. [13, 55, 203]) the ADM formalism is often written in the metric formulation. To be able to compare to these sources we can use equation (5.13) to translate the canonical momenta of the spatial vielbein to the canonical momenta of the metric.

$$\Pi^{mn}(g) = \frac{1}{2} e_a^{(m}\Pi^{n)a}(e) \tag{5.13}$$

The scalar momenta $\Pi^{RS}(M)$ follow from the variation of the Lagrangian of the form (5.14). We have to be careful to respect the symmetry of the scalar fields while taking these derivatives, but for now we may take this, together with the fundamental Poisson bracket of equation (5.15) as a definition — we will discuss this point in detail in chapter 6 when we will go through the Legendre transformation of ExFT in great detail.

$$\delta\mathcal{L}_{5\mathrm{D}} = \frac{1}{2}\, \Pi^{MN}(M)\;\; \delta\dot{M}_{MN}\, + \dots \tag{5.14}$$
$$\{M_{MN}(x), \Pi^{PQ}(M)(y)\} = \left(\delta^P_M\, \delta^Q_N + \delta^P_N\, \delta^Q_M\right) \delta^{(4)}(x-y) \tag{5.15}$$

Perhaps the most important fact we learn by analysing this theory concerns the canonical momenta of the generalised one-forms (5.9). $\Pi^l_T(A)$ are perfectly fine canonical momenta, however if we were to continue the canonical formulation with these momenta we would find that the Hamiltonian takes a rather undesirable form and the following canonical analysis is quite unappealing. It turns out that this is telling us something important. The canonical momenta that the theory seems to want us to use are the $\Pi^l_T(A)$ given by equation (5.16). We subtract the topological term contribution from the generic canonical momenta $\Pi^l_T(A)$ and arrive at the $P^l_T(A)$ — which consequentially appear as if the topological term does not exist. Writing the canonical Hamiltonian in terms of the new variables gives a greatly simplified appearance and also simplifies the gauge transformations a great deal.

$$
\begin{aligned}
P^l_L(A) :=& \Pi^l_L(A) - 4\kappa_5\, \epsilon^{lmnr} d_{MNL}\, A^M_m\, F^N_{nr} &(5.16)\\
=& \frac{e}{N} g^{ln}\, M_{LN}\, \left[F^N_{tn} + N^k F^N_{nk}\right] &(5.17)
\end{aligned}
$$

Of course this redefinition does not actually remove the complication coming from the topological term and in some sense it may just be moving it to another part of the analysis. When we are working with $P^l_L(A)$ we are no longer working with the actual canonical momenta and so the Poisson bracket structure is affected since this redefinition is in fact not a canonical transformation. The Poisson bracket of the new momenta with themselves is no longer vanishing $\{P^l_L(A), P^k_K(A)\} \neq 0$ since the topological contribution, which we subtracted, depends on the one-forms. We can explicitly compute this Poisson bracket and we find that it is given by the rather uninspiring expression of equation (5.18). Here the upper letter at the derivative operator indicates the coordinate along which the derivative acts.

$$
\begin{aligned}
\{P^l_L(A)(x), P^k_K(A)(y)\} =& + 8\kappa_5\, \epsilon^{klrs}\, d_{LKM}\, \partial^y_r A^M_s(y)\, \delta(x-y)\\
&+ 8\kappa_5\, \epsilon^{klrs}\, d_{LKM}\, A^M_r(y)\, \partial^x_s \delta(x-y)\\
&+ 8\kappa_5\, \epsilon^{klrs}\, d_{LKM}\, \partial^x_r A^M_s(x)\, \delta(x-y)\\
&+ 8\kappa_5\, \epsilon^{klrs}\, d_{LKM}\, A^M_r(x)\, \partial^x_s \delta(x-y) \qquad (5.18)
\end{aligned}
$$

In the following analysis — instead of using this equation, although it is completely correct — we carry out the calculations in a sort of exact perturbation approach in orders of the coefficient of the topological term κ_5. First we compute the result for $\kappa_5 = 0$ and then we improve on it by calculating the terms linear in κ_5. This proves to be a good way to manage this complication.

Even though $P^l_L(A)$ introduces this new difficulty it seems that the greatly simplified Hamiltonian is worth this trade off. Moreover we find in section 5.3.2 that the new momenta have rather nice transformation properties, which we may also take as a hint from the theory that these are the best variables to use in the canonical formulation.

5.2.2 Canonical Hamiltonian

Now that we have all the canonical momenta we can proceed to calculate the canonical Hamiltonian by writing the ADM decomposition of all terms in the Lagrangian as described in section 4.2 and then carrying out the Legendre transformation as described in section 4.1. In this section we will simply state the resulting Hamiltonian and refer the reader to the detailed discussion regarding the Legendre transformation of ExFT in chapter 6 which includes all the terms and intricacies that appear in the five-dimensional supergravity.

The resulting canonical Hamiltonian density is given by equation (5.19). The fields whose conjugate canonical momenta we found to be primary constraints of shift type are factored out in the Hamiltonian (5.19) since we expect their cofactors to become secondary constraints. We will see that these secondary constraints will generate time evolution, spatial diffeomorphisms and $\mathrm{U}(1)^{27}$ gauge transformations respectively.

$$
\begin{aligned}
\mathcal{H}_{5\mathrm{D}} = &+ N \cdot \Big[+ \frac{1}{4e}\Pi_{ab}(e)\ \Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\ R \\
&\qquad + \frac{3}{2e}\Pi^{MN}(M)\ \Pi^{RS}(M)\ M_{MR}\ M_{NS} - \frac{e}{24}g^{kl}\ \partial_k M_{MN}\ \partial_l M^{MN} \\
&\qquad + \frac{e}{4}M_{MN}\ g^{rm}\ g^{sn}\ F^M_{rs}\ F^N_{mn} + \frac{1}{2e}g_{lm}\ M^{KL}\ P^l_L\ P^m_K \Big] \\
&+ N^n \cdot \Big[+ 2\ \Pi^m{}_a(e)\ \partial_{[n}e_{m]a} - e_{na}\ \partial_m \Pi^m{}_a(e) \\
&\qquad + \frac{1}{2}\Pi^{MN}(M)\ \partial_n M_{MN} \\
&\qquad + F^M_{nl} P^l_M \Big] \\
&+ A^M_t \cdot \Big[- \partial_l P^l_M - 3\kappa_5\, \epsilon^{lmnr}\ d_{MNP}\ F^N_{lm}\ F^P_{nr} \Big] \\
&+ \dot{N} \cdot \Pi(N) + \dot{N}_a \cdot \Pi^a(N_a) + \dot{A}^M_t \cdot \Pi_M(A_t)
\end{aligned}
\tag{5.19}
$$

The terms that appear in the ADM formulation of canonical general relativity are simply the first and fourth line of this Hamiltonian. One should note here that the two quadratic terms of the vielbein momenta are distinct since the are contracted differently. R here simply denotes the four-dimensional spatial Ricci scalar.

Regarding the modified momenta $P^l_M(A)$ we find that there are just three terms in the Hamiltonian that depend on these variables. Furthermore there is only a single topological term that explicitly depends on the topological coefficient κ_5 — in analogy to the topological term of the Lagrangian, albeit of the form $F \wedge F$. One can contrast this simplified form of the Hamiltonian to the generic form by reinserting the definition (5.16). This comparison will prove much more striking for the case of the exceptional field theory, further strengthening the argument for using the canonical momenta $P^l_M(A)$.

The last line of terms in the Hamiltonian (5.19) come from the Legendre transformations of the fields that do not appear with time derivatives in the Lagrangian and hence they each contain as factors momenta that are primary constraints. Since we do not want to restrict the canonical analysis to the primary constraint surface we keep these terms intact — however we find that these terms do not have much effect on the further analysis.

5.2.3 Primary constraints

In order to construct the full set of canonical constraints we have to find the complete set of primary constraints first. When calculating the canonical momenta in section 5.2.1 we have already identified a number of shift type primary constraints that are immediately apparent. There are six further primary constraints that follow from the fact that the canonical momenta of the spatial vielbein (5.8) only depend on the symmetric part of the anholonomy. We call these the *Lorentz constraints* L_{ab}, written as in equation (5.23), with $L_{ab} = L_{[ab]}$ since they are associated to the Lorentz symmetry that is manifest because we are using the vielbein formalism. In total we count 38 primary constraints and the complete set is as follows.

$$\Pi(N) = 0. \tag{5.20}$$
$$\Pi^a(N_a) = 0 \tag{5.21}$$
$$\Pi_M(A_t^M) = 0 \tag{5.22}$$
$$L_{ab} := e_{m[a}\Pi^m{}_{b]}(e) = 0 \tag{5.23}$$

5.2.4 Fundamental Poisson brackets

Before we can construct the full set of canonical constraints we need to define the fundamental Poisson bracket structure. The fundamental non-vanishing equal time Poisson brackets are as follows.

$$\{N(x), \Pi(N)(y)\} = \delta^{(4)}(x-y) \tag{5.24}$$
$$\{N^n(x), \Pi_m(N^k)(y)\} = \delta^n_m \delta^{(4)}(x-y) \tag{5.25}$$
$$\{e_n{}^a(x), \Pi^m{}_b(e)(y)\} = \delta^m_n \delta^a_b \delta^{(4)}(x-y) \tag{5.26}$$
$$\{A_t^M(x), \Pi_N(A_t^K)(y)\} = \delta^M_N \delta^{(4)}(x-y) \tag{5.27}$$
$$\{A_m^M(x), \Pi^n_N(A_k^K)(y)\} = \{A_m^M(x), P^n_N(y)\} = \delta^M_N \delta^n_m \delta^{(4)}(x-y) \tag{5.28}$$
$$\{M_{MN}(x), \Pi^{PQ}(M)(y)\} = \left(\delta^P_M\, \delta^Q_N + \delta^P_N\, \delta^Q_M\right) \delta^{(4)}(x-y) \tag{5.29}$$

As expected the redefined one-form momentum (5.16) does not modify the Poisson bracket (5.28) since the topological contribution only depends on the one-form itself. We have to remember however that $\{P^l_L(A), P^k_K(A)\} \neq 0$.

It is worth repeating here that due to the use of the implicit formalism for the coset constraints — as explained in section 4.3 — the relation (5.29) is simply the fundamental Poisson bracket of a generic scalar matrix and there is not coset projector term.

These fundamental Poisson brackets — together with the linearity, Leibniz and antisymmetry properties of the Poisson bracket — are all that is needed to evaluate any Poisson bracket. There are however a number of very useful derived identities that can be constructed from the fundamental brackets and we list some useful Poisson bracket identities, that we need to use in the following computations, in the appendix B.

5.2.5 Total Hamiltonian and secondary constraints

Having found the set of primary constraints we now need to verify their consistency and in the process new secondary constraints emerge whose consistency we then also verify as explained in section 4.1.

The total Hamiltonian (5.30) is found by adding the primary constraints with arbitrary phase space coefficient functions C_0, C_1, C_2, C_3 to the canonical Hamiltonian — we are free to do so since we can arbitrarily extend the Hamiltonian away from the primary constraint surface.

$$\mathcal{H}_T := \mathcal{H}_{5\mathrm{D}} + C_0 \cdot \Pi(N) + (C_1)_a \cdot \Pi^a(N_a) + (C_2)^M \cdot \Pi_M(A_t) + (C_3)^{ab} \cdot L_{ab} \quad (5.30)$$

For some primary constraint Φ to be consistent for all times we need its total time evolution to vanish (5.31).

$$\dot{\Phi} = \{\Phi, \mathcal{H}_T\} \stackrel{!}{=} 0 \quad (5.31)$$

A useful observation that can be made about the set of primary constraints that we have found, is that they all Poisson-commute amongst each other, with the exception of the Lorentz constraints $\{L_{ab}, L_{cd}\} \neq 0$. This should be expected because we want them to form the Lorentz subalgebra of the gauge algebra.

The shift type primary constraints each lead to a secondary constraint since their conjugated fields necessarily appear as Lagrange multipliers. The conservation of the lapse function momentum constraint leads to the *Hamilton* constraint (5.32) $H_{\mathrm{Ham}} := \{\mathcal{H}_{5\mathrm{D}}, \Pi(N)\}$. Likewise we find the *diffeomorphism* constraint (5.33) $(H_{\mathrm{Diff}})_n := \{\mathcal{H}_{5\mathrm{D}}, \Pi_a(N_a)\}\, e_n^a$ and the *Gauß* constraint (5.34) takes the form of the transformation $(H_{\mathrm{Gauß}})_M := \{\mathcal{H}_{5\mathrm{D}}, \Pi_M(A_t^M)\}$. We have chosen to insert a vielbein contraction into the definition of the diffeomorphism constraint as it is advantageous to have a curved index on the gauge parameter. The name Gauß constraint was chosen in analogy to Maxwell theory and Gauß's law that appears there. We will see that the constraints to indeed generate the transformations that their names suggest.

$$\begin{aligned}
H_{\mathrm{Ham}} =& + \frac{1}{4e}\Pi_{ab}(e)\ \Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\ R \\
&+ \frac{3}{2e}\Pi^{MN}(M)\ \Pi^{RS}(M)\ M_{MR}\ M_{NS} - \frac{e}{24}g^{kl}\ \partial_k M_{MN}\ \partial_l M^{MN} \\
&+ \frac{e}{4}M_{MN}\ g^{rm}\ g^{sn}\ F^M_{rs}\ F^N_{mn} + \frac{1}{2e}g_{lm}\ M^{KL}\ P^l_L\ P^m_K && (5.32)\\
(H_{\mathrm{Diff}})_n =& + 2\ \Pi^m{}_a(e)\ \partial_{[n}e_{m]a} - e_{na}\ \partial_m \Pi^m{}_a(e) \\
&+ \frac{1}{2}\Pi^{MN}(M)\ \partial_n M_{MN} \\
&+ F^M_{nl}P^l_M && (5.33)\\
(H_{\mathrm{Gauß}})_M =& - \partial_l P^l_M - 3\kappa_5\, \epsilon^{lmnr}\ d_{MNP}\ F^N_{lm}\ F^P_{nr} && (5.34)
\end{aligned}$$

The secondary constraints are generated by the time evolution of the primary constraints and as such they restrict the canonical coordinates dynamically. We can see that the Gauß constraints $H_{\mathrm{Gauß}}$ only depend on the gauge field and its momentum and take the form of the constraint that we would expect in Maxwell theory plus a topological contribution.

So far we have only discussed the shift type primary constraints, but we still need to discuss the Lorentz constraints. The first line of the diffeomorphism constraint H_{Diff} concerns general relativity and we can rewrite these terms as in equation (5.35). This rewriting reveals a Lorentz constraint term with the coefficient given by the spin connection $\omega_n{}^{ab} = e^{ak}\nabla_n e_k{}^b$ which is field dependent. Here we also make use of the covariant derivative ∇_n that contains the Levi-Civita connection and the covariant derivative D_m that contains the spin connection.

$$+2\ \Pi^m{}_a(e)\ \partial_{[n}e_{m]a} - e_{na}\ \partial_m\Pi^m{}_a(e) = -e_n{}^a\, D_m\Pi^m{}_a(e) + \omega_n{}^{ab}\, L_{ab} \quad (5.35)$$

The existence of this term tells us that the diffeomorphism constraint and the Lorentz constraint do not Poisson-commute and we might want to try to redefine the diffeomorphism constraint as in equation (5.36) in order to make them commute.

$$(\tilde{H}_{\text{Diff}})_n := (H_{\text{Diff}})_n - \omega_n{}^{ab}\, L_{ab} \tag{5.36}$$

This redefinition would essentially be a different choice of basis in the algebra of constraints — but we will find that the constraint (5.33) generates the expected gauge transformations (cf. equation (5.55)) and is easier to work with in general. Hence we continue to work with (5.33).
If we now investigate the consistency of the Lorentz constraints we find that no new secondary constraints are generated, however consistency requires that the coefficient function $(C_3)^{ab}$ of the Lorentz constraint term $(C_3)^{ab}\, L_{ab}$ in the total Hamiltonian has to be $(C_3)^{ab} = -N^n\, \omega_n{}^{ab}$ for the constraints to be consistent. This term is precisely the same term we identified in equation (5.36) — implying that the total Hamiltonian overall does not contain a Lorentz constraint term. We will see in section 5.3.4 that because the Lorentz constraints are first class they will explicitly appear in the extended Hamiltonian nonetheless.

The final total Hamiltonian can be written as (5.37) with the arbitrary coefficients C_0, $(C_1)_a$ and $(C_2)^M$.

$$\begin{aligned}\mathcal{H}_{\text{T}} \;=&+ N\cdot H_{\text{Ham}} + N^n\cdot (H_{\text{Diff}})_n + A_t^M\cdot (H_{\text{Gauß}})_M \\ &+ C_0\cdot \Pi(N) + (C_1)_a\cdot \Pi^a(N_a) + (C_2)^M\cdot \Pi_M(A_t) \\ &- N^n\, \omega_n{}^{ab}\, L_{ab}\end{aligned} \tag{5.37}$$

The total time evolution of a generic phase space function F can be written as the weak equality (5.38) and is equivalent to the Lagrangian time evolution [28].

$$\dot{F} \approx \{F, \mathcal{H}_T\} \tag{5.38}$$

When we construct the extended Hamiltonian in section 5.3.4 we find that the exact coefficients in the total Hamiltonian are irrelevant since they are replaced in the extended Hamiltonian by arbitrary coefficient functions. If one wants to explicitly calculate the time evolution of some phase space function it is in practice easiest to use the Leibniz rule of the Poisson bracket and then insert the gauge transformations that we compute in section 5.3.2.

Calculating the total time evolution of the secondary constraints we do not find any new tertiary constraints. The set of constraints we found thus far is hence complete and consistent. We do not expect to find a constraint associated to the exceptional $\text{E}_{6(6)}(\mathbb{R})$ symmetry of the theory since it is a global symmetry and therefore not generated by a canonical constraint.

We find that the Hamiltonian (5.19) is given entirely in terms of constraints and therefore weakly vanishing $\mathcal{H}_{\text{5D}} \approx 0$, meaning that the time evolution can be written entirely in terms of gauge transformations. This observation is a general fact of generally covariant theories and in particular true for general relativity [28]. This theory inherits this property from general relativity because supergravity is built upon it.

In the following canonical analysis we will make use of integrated of smeared constraints — as explained in section 4.1 — in order to avoid dealing with derivatives of Dirac delta distributions. Therefore we take all the constraints to be integrated over the spatial hypersurface with appropriate tensors of gauge parameters that are test functions. Concerning the notation, we distinguish the smeared constraints by adjoining the name of the gauge parameter to the constraint name. For example, the smeared diffeomorphism constraint $H_{\text{Diff}}[\xi]$ is defined by equation (5.39), the gauge parameter $\xi^n(x)$ is a spatial four-vector of test functions defined on the spatial hypersurface.

$$H_{\text{Diff}}[\xi] \ := \ \int (H_{\text{Diff}})_n(x)\ \xi^n(x)\ d^4x \tag{5.39}$$

Furthermore we choose not to insert any symmetry factors in the smeared constraints. In this theory this only concerns the Lorentz constraints and their integrated form is given by equation (5.40), with an antisymmetric gauge parameter $\gamma^{ab} = \gamma^{[ab]}$.

$$L[\gamma] \ := \ \int L_{ab}(x)\ \gamma^{ab}(x)\ d^4x \tag{5.40}$$

5.3 Canonical analysis: Gauge transformations and gauge algebra

In the previous section 5.2 we have constructed the Hamiltonian formulation of the theory and found the complete set of canonical constraints. In this section we analyse this canonical formulation by computing the full set of gauge transformations generated by the constraints, furthermore we compute the full gauge algebra that the constraints form with the Poisson bracket. This analysis allows us to determine the number of physical degrees of freedom and to construct the extended Hamiltonian.

5.3.1 Diffeomorphism weight and the Lie derivative

Before we begin with the analysis we briefly review the Lie derivative of standard diffeomorphisms and the diffeomorphism weight.

We write the standard Lie derivative of a tensor T with diffeomorphism parameter ξ as $\mathcal{L}_\xi T$. And we define the diffeomorphism weight $\Lambda(T)$ as the coefficient of the weight term in the Lie derivative. The Lie derivative of a vector T, for example, would be written as in equation (5.41) with the diffeomorphism weight denoted by Λ.

$$(\mathcal{L}_\xi T)^\nu = \underbrace{\xi^\mu\,\partial_\mu T^\nu}_{\text{transport term}} - \underbrace{\partial_\mu \xi^\nu\, T^\mu}_{\text{rotation term}} + \underbrace{\Lambda \cdot \partial_\mu \xi^\mu}_{\text{weight term}} \tag{5.41}$$

The diffeomorphism weights of the canonical variables are listed in table 5.1. Since the vielbein determinant e is a tensor density, and since it appears as a factor in every kinetic term in the action, the momenta all have weight one too.

Object	Weight Λ
$e_m{}^a$, A_m^M, M_{MN}	0
$\Pi^m{}_a(e)$, $P_M^m(A)$, $\Pi^{RS}(M)$	1
e	1

Table 5.1: The standard diffeomorphism weights of the canonical coordinates of bosonic maximal supergravity in five dimensions.

In the following the Lie derivatives always include the correct weight terms and we have to pay attention to this fact during the computations.

5.3.2 Gauge transformations

In this section we explicitly compute all the infinitesimal gauge transformations $\delta_\lambda X$, here λ is a gauge parameter and X represents a canonical coordinate. Canonically we think of this as $\delta_\lambda X = \{X, \mathcal{H}[\lambda]\}$ where $\mathcal{H}[\lambda]$ represents a smeared constraint. The Poisson brackets are then evaluated using the fundamental brackets defined in section 5.2.4 and using the identities listed in the appendix B. In general we will in the following only list the non-vanishing gauge transformations unless we want to make a point of a specific transformation vanishing.
For the sake of simplicity we omit the notation of the coordinate dependence. In the following expressions the gauge parameters on the right hand side of each gauge transformation only depend on the coordinate of the field that the smeared constraint acts upon.

First we look at the primary constraints of shift type and as expected we find that they generate shift transformations on the fields canonically conjugate to the vanishing momenta.

$$\{N, \Pi(N)[\lambda_1]\} = \lambda_1 \tag{5.42}$$
$$\{N_a, \Pi(N_b)[\lambda_2]\} = (\lambda_2)_a \tag{5.43}$$
$$\{A_t^N, \Pi(A_t^M)[\lambda_3]\} = (\lambda_3)^N \tag{5.44}$$

The Lorentz constraints generate local Lorentz transformations on the spatial vielbein and its conjugate momentum. The Lorentz transformations take the form of rotations of the flat Lorentz indices by the matrix of gauge parameters. All quantities that can be expressed purely via the metric tensor are invariant under these transformations and hence also the vielbein determinant is Lorentz invariant.

$$\{e_n{}^a, L[\gamma]\} = +\, e_{nb}\, \gamma^{ba} \tag{5.45}$$
$$\{\Pi^n{}_a(e), L[\gamma]\} = +\, \Pi^n{}_c(e)\, \gamma^{cb}\, \delta_{ba} \tag{5.46}$$

In the canonical formalism time evolution can be thought of as a gauge transformation [28]. This time evolution is generated by the Hamilton constraint. In general relativity and supergravity, this time evolution is distinct from the time evolution that the canonical Hamiltonian generates because the Hamilton constraint is just one constituent constraint of the canonical Hamiltonian. The meaning of this is that time evolution is not unique in a gauge theory and there are always other possible time evolutions that differ by gauge transformations and this gauge freedom is not captured by the Hamilton constraint. Conversely the Hamiltonian time evolution without the Hamilton constraint is just gauge transformations and there is no real time evolution at all.

For theories that are second order in the time derivatives of the fields, the Hamilton constraint should contain terms that are quadratic in the conjugate momenta. This fact then implies that the time evolution of the fields is given by terms linear in the conjugate canonical momenta — and this is in fact what we find below. The time evolution of the canonical momenta themselves is more subtle, it captures most of the dynamics and depends on the theory at hand. In the case of general relativity and supergravity we find the time components of the Einstein and Maxwell equations in these transformations.

The metric contracts all terms besides the topological term in the Lagrangian and therefore we expect the vielbein canonical momentum $\Pi^n{}_a(e)$ to have a complicated time evolution. Therefore to simplify the complicated computation of (5.48) it makes sense to first compute the time evolution of the spatial Einstein-Hilbert term $e\,R_4$, as given in the appendix in equation (B.2), on its own. Most of the identities following (B.2) also have to be used in the computation of (5.48). The spatial Einstein equation, in the vielbein form, that we can see in the third line of (5.48) comes precisely from the time evolution of the spatial Einstein-Hilbert term $e\,R_4$.

The equation (5.13) relating the vielbein momenta to the spatial metric momenta allows us to compare the time evolution of $\Pi^n{}_a(e)$ to the time evolution of the canonical momentum of the metric in pure general relativity, as given in the references [55, 203].

$$\{e_{na}, H_{\mathrm{Ham}}[\phi]\} = + \frac{\phi}{2e}\, g_{mn}\, \Pi^m{}_a(e) - \frac{\phi}{6e}\, \Pi(e)\, e_{na} \tag{5.47}$$

$$\begin{aligned}\{\Pi^n{}_a(e), H_{\mathrm{Ham}}[\phi]\} = &+ \frac{\phi}{4e}\, \Pi_{bc}(e)\, \Pi_{bc}(e)\, e_a{}^n - \frac{\phi}{2e}\Pi^k{}_b(e)\, \Pi^n{}_b(e)\, e_{ka} \\
&- \frac{\phi}{12e}\, \Pi^2(e)\, e_a{}^n + \frac{\phi}{6e}\, \Pi(e)\, \Pi^n{}_a(e) \\
&- 2\phi e\left(R^{nk}\, e_{ka} - \frac{1}{2}R\, e_a{}^n\right) \\
&+ 2e\left(\nabla_a\nabla^n\phi - \nabla^k\nabla_k\phi\, e_a{}^n\right) \\
&+ \frac{3\phi}{2e}\Pi^{MN}(M)\, \Pi^{RS}(M)\, M_{MR}\, M_{NS}\, e_a{}^n \\
&+ \frac{\phi e}{24}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{kl}\, e_a{}^n - \frac{\phi e}{12}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{ln}\, e_a{}^k \\
&- \frac{\phi e}{4} M_{MN}\, F^M_{rs}\, F^N_{km}\, g^{rk}\, g^{sm}\, e_a{}^n + \phi e\, M_{MN}\, F^M_{rs}\, F^N_{km}\, g^{rk}\, g^{mn}\, e_a{}^s \\
&+ \frac{\phi}{2e} M^{KL}\, P^l_L(A)\, P^k_K(A)\, g_{lk}\, e_a{}^n - \frac{\phi}{e} M^{KL}\, P^n_K(A)\, P^l_L(A)\, e_{la}\end{aligned} \tag{5.48}$$

It is a general fact that the Hamiltonian generates the time derivative of the field that it acts on in the Poisson bracket $\{X, \mathcal{H}\} = \dot{X}$. In generally covariant theories this is already true for the Hamilton constraint because the other transformations are "actual" gauge transformations. The time evolution of the scalar fields (5.49) and of the one-form fields (5.51) is given by their canonical momenta, which are proportional to the time derivatives of the fields. Therefore the time evolution of their canonical momenta (5.50) and (5.52) is equivalent to the Euler-Lagrange equations of the Lagrangian for these fields, because the time derivative of the canonical momenta is proportional to the second time derivative of the fields.

$$\begin{aligned}
\{M_{MN}, H_{\text{Ham}}[\phi]\} &= + \frac{6}{e}\phi\ \Pi^{QP}(x)\ M_{MQ}\ M_{NP} && (5.49)\\
\{\Pi^{MN}(M), H_{\text{Ham}}[\phi]\} &= - \partial_l \left(\frac{\phi e}{6}\ g^{kl}\ \partial_k M^{MN} \right) \\
&\quad - \frac{\phi e}{6}\ g^{kl}\ \partial_k M_{KL}\ \partial_l M^{KM}\ M^{LN} \\
&\quad - \frac{6\phi}{e}\ \Pi^{PM}(M)\ \Pi^{NR}(M)\ M_{PR} \\
&\quad - \frac{\phi e}{2}\ g^{rm}\ g^{sn}\ F^{M}_{rs}\ F^{N}_{mn} \\
&\quad + \frac{\phi}{e} g_{lm}\ P^l_L\ P^m_K\ M^{KM}\ M^{LN} && (5.50)\\
\{A^N_n, H_{\text{Ham}}[\phi]\} &= + \frac{\phi}{e}\ g_{nl}\ M^{NL}\ P^l_L && (5.51)\\
\{P^l_S, H_{\text{Ham}}[\phi]\} &= + \partial_m \left(e\,\phi\ M_{NS}\ g^{rm}\ g^{ls}\ F^N_{rs} \right) && (5.52)\\
&\quad - \frac{12\kappa_5\phi}{e}\ g_{mk}\ M^{KL}\ d_{SLM}\ \epsilon^{lmrs}\ F^M_{rs}\ P^k_K
\end{aligned}$$

The Gauß constraint generates abelian $\mathrm{U}(1)^{27}$ gauge transformations on the one-form gauge field A^N_n (5.53). As we can see from equation (5.54) the modified momentum P^n_N from (5.16) is invariant under the $\mathrm{U}(1)^{27}$ gauge transformations — like it would be in the free theory. This is a nice property of the modified momentum and further evidence that this is the better canonical variable to use. The Gauß constraint is independent of the other fields and so they do not transform. In the analysis of the exceptional field theory in chapter 6 we find that this behaviour is drastically modified as the one-form fields are used as a gauge connection for the exceptional generalised diffeomorphisms and they are found to be generated by the analogue of the Gauß constraint — i.e. the cofactor of the Lagrange multiplier A^M_t.

$$\begin{aligned}
\{A^N_n, H_{\text{Gauß}}[\zeta]\} &= + \partial_n \zeta^N && (5.53)\\
\{P^n_N, H_{\text{Gauß}}[\zeta]\} &= 0 && (5.54)
\end{aligned}$$

The diffeomorphisms on the spatial hypersurfaces of the ADM decomposition of the space-time are generated by the diffeomorphism constraint. We can express these diffeomorphisms via the standard Lie derivative (including the appropriate weight terms as given in table 5.3.1). This way we can see that the redefined diffeomorphism constraint of equation (5.36) would contribute additional terms for the transformation of the vielbein and its conjugate momentum, which we do not want.

$$\begin{aligned}
\{e_n{}^a, H_{\text{Diff}}[\xi]\} &= + \mathcal{L}_\xi e_n{}^a && (5.55)\\
\{\Pi^n{}_a(e), H_{\text{Diff}}[\xi]\} &= + \mathcal{L}_\xi \Pi^n{}_a(e) && (5.56)\\
\{M_{MN}, H_{\text{Diff}}[\xi]\} &= + \mathcal{L}_\xi M_{MN} && (5.57)\\
\{\Pi^{MN}(M), H_{\text{Diff}}[\xi]\} &= + \mathcal{L}_\xi \Pi^{MN}(M) && (5.58)\\
\{A^N_n, H_{\text{Diff}}[\xi]\} &= - \xi^m\ F^N_{nm} && (5.59)\\
&= + \mathcal{L}_\xi A^N_n + \delta_{(\xi^m A_m)} A^N_n && (5.60)\\
\{P^l_S, H_{\text{Diff}}[\xi]\} &= + \mathcal{L}_\xi P^l_S + \xi^l\ (H_{\text{Gauß}})_S && (5.61)\\
&\approx + \mathcal{L}_\xi P^l_S
\end{aligned}$$

We should pay attention to the additional terms in the transformations of the one-forms (5.60) and their conjugate momenta (5.61). These extra terms are due to the parametrisation of the Lagrangian. For the one-forms (5.60) we find that the transformation is a Lie derivative only up to a $\mathrm{U}(1)^{27}$ gauge transformation. This gauge transformation, generated by the constraint $H_{\text{Gauß}}$, is understood to be read as $\delta_{(\xi^m A_m)} A_n^N = \{A_n^N, H_{\text{Gauß}}[\xi^m A_m^M]\} = +\partial_n(\xi^m A_m^N)$. The transformation of the canonical momenta P_S^l is a Lie derivative only up to a Gauß constraint $H_{\text{Gauß}}$ term and therefore it is weakly equal to the pure Lie derivative term.

The shortest computation of (5.61) involves working order by order in κ_5, but also makes use of the Leibniz rule and the Schouten identity from appendix A. We want to briefly sketch this way of doing the calculation since it is advantageous to proceed in the same fashion in this calculation for exceptional field theory when there are many more additional complications.
To do so we define Δ_T^l to be the topological term — i.e. linear in κ_5 — in the momentum $\Delta_T^l(A) := P_T^l(A) - \Pi_T^l(A)$. We proceed by using the Leibniz property of the Poisson bracket in the very first step of the computation (5.62), this will guarantee the right structure of terms in the following steps. Then we use the definition of the modified momentum to arrive at (5.63).

$$
\begin{aligned}
&\{P_Q^q(x), \int d^4y\, \xi^n(y)\, F_{nl}^T(y)\, P_T^l(y)\} \\
&= \int d^4y\, \xi^n(y)\, \{P_Q^q(x), F_{nl}^T(y)\}\, P_T^l(y) + \int d^4y\, \xi^n(y)\, \{P_Q^q(x), P_T^l(y)\}\, F_{nl}^T(y) \qquad (5.62)\\
&= \int d^4y\, \xi^n(y)\, \{\Pi_Q^q(x), F_{nl}^T(y)\}\, P_T^l(y) + \int d^4y\, \xi^n(y)\, \{\Pi_Q^q(x), \Delta_T^l(y)\}\, F_{nl}^T(y) \\
&\qquad + \int d^4y\, \xi^n(y)\, \{\Delta_Q^q(x), \Pi_T^l(y)\}\, F_{nl}^T(y) \qquad (5.63)
\end{aligned}
$$

The first term in (5.63) can be written as the Lie derivative plus the pure Maxwell theory Gauß constraint but all in terms of the modified momentum P_Q^q.

$$
\int d^4y\, \xi^n(y)\, \{\Pi_Q^q(x), F_{nl}^T(y)\}\, P_T^l(y) = \mathcal{L}_\xi P_Q^q + \xi^q\,(-\partial_n P_Q^n) \qquad (5.64)
$$

Using the Dirac delta distribution identity from the appendix A.2 we can evaluate both of the remaining brackets in (5.63). We find that we can write each of these brackets as three terms and taken together four terms cancel while the other two join and we arrive at equation (5.65). Using the Schouten identity we can rewrite this as (5.66) if we also realise that we can use the symmetry of the $\mathrm{E}_{6(6)}$ invariant symbol to avoid generating more terms in the Schouten identity.

$$
\begin{aligned}
&\int d^4y\, \xi^n(y)\, \{\Pi_Q^q(x), \Delta_T^l(y)\}\, F_{nl}^T(y) + \int d^4y\, \xi^n(y)\, \{\Delta_Q^q(x), \Pi_T^l(y)\}\, F_{nl}^T(y) \\
&= -12\, \kappa_5\, \epsilon^{q n_1 n_2 n_3}\, \xi^l\, d_{MNQ}\, F_{l n_1}^M\, F_{n_2 n_3}^N \qquad (5.65)\\
&= -3\, \kappa_5\, \epsilon^{n_1 n_2 n_3 n_4}\, \xi^q\, d_{MNQ}\, F_{n_1 n_2}^M\, F_{n_3 n_4}^N \qquad (5.66)
\end{aligned}
$$

Taking the results from (5.64) and (5.66) together we arrive at the equation (5.61) that we are looking for.

5.3.3 Algebra of constraints

Having computed how all the constraints act on the canonical coordinates in section 5.3.2, we can now use this knowledge to compute the algebra that the canonical constraints generate under the Poisson bracket by acting on each other.
The interpretation of an integrated constraint $\mathcal{H}[\lambda]$ acting with the Poisson bracket on a canonical coordinate X from the right $\delta_{\mathcal{H}[\lambda]}X = \{X, \mathcal{H}[\lambda]\}$ was that of an infinitesimal gauge transformation $\delta_{\mathcal{H}[\lambda]}X$. Consequentially the interpretation of a Poisson bracket of two constraints $\{\mathcal{H}_1[\xi], \mathcal{H}_2[\lambda]\}$ is that of a commutator of two infinitesimal gauge transformations. To prove this we can start with the commutator of two infinitesimal gauge transformations $[\delta_{\mathcal{H}_2[\lambda]}, \delta_{\mathcal{H}_1[\xi]}]X$ and expand this expression using the definition the commutator. Writing the infinitesimal transformations in terms of Poisson brackets we can rewrite this using the antisymmetry and the Jacobi identity of the Poisson bracket to arrive at equation (5.67). The different ordering of the transformations comes from the fact that the constraints act from the right whereas the gauge transformations act from the left.

$$[\delta_{\mathcal{H}_2[\lambda]}, \delta_{\mathcal{H}_1[\xi]}]X = \{X, \{\mathcal{H}_1[\xi], \mathcal{H}_2[\lambda]\}\} \tag{5.67}$$

Concerning the actual computation of the algebra it is often easiest to take apart the constraint with fewer terms and then apply the gauge transformations that we have found. If the diffeomorphism constraint is involved one should make use of the fact that most fields transform as a Lie derivative, this fact greatly simplifies the calculation.

The shift type primary constraints Poisson-commute with every other constraint and hence all algebra relations concerning them are vanishing.

Below we state the full algebra of the canonical constraints.

$$\begin{aligned}
\{H_{\mathrm{Ham}}[\theta], H_{\mathrm{Ham}}[\tau]\} &= H_{\mathrm{Diff}}[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}] \\
&\qquad - L\,[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}\,\omega_{nab}] && (5.68)\\
\{H_{\mathrm{Diff}}[\lambda], H_{\mathrm{Ham}}[\theta]\} &= H_{\mathrm{Ham}}[\mathcal{L}_\lambda\theta] + H_{\mathrm{Gauß}}\left[\frac{\theta}{e}\lambda^p\,g_{pk}\,P^k_L\,M^{LM}\right] && (5.69)\\
\{H_{\mathrm{Ham}}[\theta], H_{\mathrm{Gauß}}[\xi]\} &= 0 && (5.70)\\
\{H_{\mathrm{Diff}}[\lambda], H_{\mathrm{Diff}}[\rho]\} &= H_{\mathrm{Diff}}\,[[\lambda,\rho]^n] = H_{\mathrm{Diff}}\,[\mathcal{L}_\lambda\rho^n] && (5.71)\\
\{H_{\mathrm{Diff}}[\lambda], H_{\mathrm{Gauß}}[\xi]\} &= 0 && (5.72)\\
\{H_{\mathrm{Gauß}}[\xi], H_{\mathrm{Gauß}}[\zeta]\} &= 0 && (5.73)\\
\{L[\gamma], L[\rho]\} &= L[-2\gamma^{c[a}\,\rho^{b]c}] && (5.74)\\
\{H_{\mathrm{Ham}}[\phi], L[\gamma]\} &= 0 && (5.75)\\
\{H_{\mathrm{Diff}}[\lambda], L[\gamma]\} &= L[\mathcal{L}_\lambda(\gamma^{ab})] && (5.76)\\
\{H_{\mathrm{Gauß}}[\xi], L[\gamma]\} &= 0 && (5.77)
\end{aligned}$$

The relations (5.68), (5.69) and (5.71) of the algebra are also called the *universal* part of the gauge algebra — they concern the relations found in general relativity. Sometimes the algebra is restricted to the primary constraint surface where the Lorentz constraint vanishes and hence can be ignored — thus also removing the Lorentz constraint term from (5.68).
Equation (5.68) tells us that two different orderings of time evolutions by the Hamilton

constraint can only deviate from one another by a diffeomorphism and a Lorentz transformation. The time evolution is hence only unique up to these gauge transformations.

The Lorentz constraint term in equation (5.68) appears in the vielbein formalism of general relativity, depending on the choice of diffeomorphism constraint as discussed in section 5.2.5.[4] Its gauge parameter explicitly depends on the spin connection, however Lorentz invariance is unaffected by this term, because taken together the spin connection and the Lorentz constraint transform under a Lorentz transformation to cancel out the transformation of the diffeomorphism constraint.

The relation (5.69) tells us that the difference in ordering of time evolution and a diffeomorphism is essentially another time evolution where the resulting gauge parameter is the Lie derivative of the original gauge parameters. The additional Gauß constraint term, with field dependent gauge parameter, is due to the additional terms in the transformations of the one-form (5.60) and its conjugate momenta (5.61) under the diffeomorphism constraint.

We have used the term algebra to describe the above structure, however it would be more precise to refer to it as an *open-* or *pseudo*-algebra since the smearing functions on the right hand side of some relations (e.g. (5.68) and (5.69)) are field dependent [15, 28]. Commonly this distinction in terminology is not made and we will continue to refer to it as an algebra. Already for the case of pure general relativity the gauge parameter of the diffeomorphism constraint in equation (5.68) contains the inverse metric and is thus field dependent.

The gauge algebra of pure general relativity is also discussed in the references [7, 13, 55, 211]. Moreover the canonical formulation and the algebra of the canonical constraints of eleven-dimensional supergravity have been discussed in [58, 59].

The spatial diffeomorphisms, the $\mathrm{U}(1)^{27}$ gauge transformations and the Lorentz transformations each generate their own subalgebra of the overall constraint algebra, as can be seen from (5.71), (5.73) and (5.74). Since $\mathrm{U}(1)^{27}$ is an abelian Lie group the relation (5.73) vanishes since the ordering of the gauge transformations does not matter. In fact we find that the Gauß constraint commutes with every constraint.

The difference in the ordering of a diffeomorphism and a Lorentz transformation, as equation (5.76) tells us, is a Lorentz transformation with the Lie derivative acting on the Lorentz gauge parameter.[5]

In conclusion, the algebra of the canonical constraints closes with the Poisson bracket, since there are no terms that are not given by a canonical constraint. As a consequence all constraints are first class since each relation in the algebra is weakly vanishing.[6]

[4]The Lorentz constraint term in equation (5.68) can be removed by the redefinition (5.36) which is essentially a change in the basis of the algebra. Unfortunately this redefinition has numerous further undesirable consequences, including the introduction of new terms in the algebra and in some gauge transformations.

[5]The redefinition (5.36) of the diffeomorphism constraint allows us to cancel the resulting Lorentz transformation of the bracket, but as described earlier there are further unintended consequences, such as changes to the gauge transformations.

[6]If we look at the full supergravity theory including Fermions this is no longer true and there will be second class constraints that necessitate the introduction of the Dirac bracket [28].

5.3.4 Extended Hamiltonian

The extended Hamiltonian, describing the most general time evolution possible of the theory, is constructed starting from the total Hamiltonian (5.37) by adding to it all first class constraints with arbitrary phase space coefficient functions. We have found that the Poisson bracket algebra closes and hence that all constraints are first class. Therefore the extended Hamiltonian (5.78) is, in this case, just a general linear combination of all constraints. The difference between the total and the extended Hamiltonian is in this case that the parameters of the secondary constraints and the Lorentz constraints become generic functions. This makes the time evolution, of a gauge-variant function, generated by the extended Hamiltonian more general than that of the Lagrangian, which was equivalent to the time evolution of the total Hamiltonian. For gauge invariant functions — i.e. observables — the time evolution of the extended Hamiltonian is of course equivalent to that of the canonical Hamiltonian.

$$\begin{aligned}\mathcal{H}_{\mathrm{E}} &= + C_{\mathrm{Ham}} \cdot H_{\mathrm{Ham}} + (C_{\mathrm{Diff}})^n \cdot (H_{\mathrm{Diff}})_n + (C_{\mathrm{Gauß}})^M \cdot (H_{\mathrm{Gauß}})_M \\ &\quad + C_0 \cdot \Pi(N) + (C_1)_a \cdot \Pi^a(N_a) + (C_2)^M \cdot \Pi_M(A_t) \\ &\quad + (C_3)^{ab} L_{ab} \end{aligned} \tag{5.78}$$

For a general phase space function F the most general time evolution we can write down is hence given by (5.79).

$$\dot{F}(q,p) \approx \{F, \mathcal{H}_E\} \tag{5.79}$$

Using the results from section 5.3.2 and the properties of the Poisson bracket we can now compute the time evolution of any phase space function with the full gauge freedom manifest.

5.3.5 Counting the degrees of freedom

The computation of the gauge algebra told us that all canonical constraints in this theory are first class. This knowledge allows us to confirm the number of physical degrees of freedom of the theory, this calculation is also a good consistency check, to make sure that we have found all constraints. To do so we list all fields and canonical constraints in table 5.2.

Fields	#	**Primary constraints**	#	**Secondary constraints**	#
N	1	$\Pi(N)$	1	Hamilton constraint	1
N_a	4	$\Pi(N_a)$	4	Diffeomorphism constraints	4
e_{ma}	16	Lorentz constraints	6	-	0
$M_{(MN)}$	42	-	0	-	0
A_t^T	27	$\Pi(A_t^T)$	27	Gauß constraints	27
A_l^T	108	-	0	-	0
Total:	**198**	**Total:**	**38**	**Total:**	**32**

TABLE 5.2: A counting of the number of fields and the number of primary and secondary canonical constraints in the bosonic sector of $E_{6(6)}(\mathbb{R})$ invariant five-dimensional supergravity. The distinction between primary and secondary constraints is irrelevant to the number of physical degrees of freedom, but illustrates where the constraints came form.

At this point all Poisson brackets have been evaluated and the implicit formalism tells us that we should reinstate the coset constraints, as explained in section 4.3. Therefore we can now consider the scalar fields and their canonical momenta to be coset representatives and should only count the 42 independent degrees of freedom that correspond to the dimension of the $\mathrm{E}_{6(6)}(\mathbb{R})/\mathrm{USp}(8)$ coset in table 5.2.

Counting the field components we find that there are a total of 198 field variables, equivalently there are $396 = 2 \cdot 198$ canonical coordinates in the phase space of this theory.

Looking at the complete set of canonical constraints we count a total of 70 canonical constraint components. They can be split into 38 primary and 32 secondary constraints as in table 5.2, although this distinction is no longer relevant, it serves to remind us where the constraints originated from. The constraints are all first class, meaning that they are all generators of gauge transformations — as we have seen, they generate shift transformations, Lorentz transformations, time evolution, spatial diffeomorphisms and $\mathrm{U}(1)^{27}$ transformations — and as such we have to count their components twice [28].

The number of physical phase space dimensions that emerge from the total 396 phase space coordinates is thus $256 = 2 \cdot (198 - 70)$. Equivalently this means that there are $128 = 198 - 70$ physical (bosonic) degrees of freedom.[7] This number agrees with the well-known result that maximal supergravity has 128 bosonic degrees of freedom [15, 19, 74, 212].

[7]Since we are considering a field theory this is the number of degrees at freedom at each point in space-time.

Chapter 6

Canonical formulation of $E_{6(6)}$ exceptional field theory

In this chapter we calculate and analyse the canonical formulation of the (bosonic) $E_{6(6)}$ exceptional field theory. This chapter is based on and closely follows the structure of parts of the publication [41].

We begin by clarifying the notation and conventions used in this chapter in section 6.1. In section 6.2 we then compute the external ADM decomposition of the ExFT Lagrangian. In section 6.3 we compute all canonical momenta and introduce some redefinitions of the canonical variables inspired by the results of chapter 5. We then identify all primary canonical constraints in section 6.4. In section 6.5 the Legendre transformation of the ExFT Lagrangian is calculated sector by sector and the canonical Hamiltonian of the ExFT is presented in section 6.6. In section 6.7 we define the fundamental Poisson brackets. The consistency algorithm of the canonical constraints is investigated in section 6.8. In section 6.9 we compute most of the (gauge) transformations generated by the canonical constraints. We then calculate some of the canonical constraint algebra relations in section 6.10 and discuss speculative results for some of the remaining brackets. In section 6.11 we introduce the internal generalised vielbein and discuss the canonical formulation in terms of these variables.

6.1 Notation and conventions

In this section we briefly clarify the notation and the conventions used in this chapter. The notation of this chapter generally agrees with the notation used in chapter 5.

The types of indices used in this chapter are listed in table 6.1. The indices t and 0 are reserved for the external curved and flat time coordinates respectively. In the spacetime split we decompose the external curved five-dimensional index as $\mu = (t, m)$ and the flat five-dimensional index decomposes as $\alpha = (0, a)$.

The constant five-dimensional Levi-Civita symbol (without any vielbein factors) is $\epsilon^{\mu\nu\rho\sigma\tau}$. The convention $\epsilon^{klmn} := \epsilon^{tklmn}$ for the four-dimensional Levi-Civita symbol is used occasionally. The convention of the Minkowski metric signature for the external geometry is $(-++++)$, which agrees with the convention used in [25]. A (curved) time derivative ∂_t, on some function X, is sometimes written as $\dot{X}$. As explained in chapter 4 we use the notation that $\Pi(X)$, with the appropriate indices, is used to denote the canonical momenta that are canonically conjugate to any fields X.

As was discussed in section 3.5 (and similarly in chapter 5), the scalar fields $M_{MN} = M_{(MN)}$ of the $E_{6(6)}$ ExFT parametrise the $E_{6(6)}/USp(8)$ coset and therefore only 42

Type of index	Dimension (real)	Letters used
Fundamental rep. of $E_{6(6)}$	27	$K, L, M, N, \dots, X, Y, Z$
Fundamental rep. of USp(8)	8	$A, B, C, D, E, \dots, J$
Curved (external)	5	$\mu, \nu, \rho, \sigma, \tau, \dots$
Curved (time)	1	t
Curved (external spatial)	4	$k, l, m, n, o, p, q, r, s, u, \dots$
Flat (external)	5	$\alpha, \beta, \gamma, \delta, \dots$
Flat (time)	1	0
Flat (external spatial)	4	$a, b, c, d, e, f, g, h, \dots$

TABLE 6.1: Conventions for the indices used in chapter 6, their dimensions and the descriptions of the types of indices.

out of the 378 components of the symmetric matrix M_{MN} are actually independent. From the canonical analysis of the SL(n)/SO(n) scalar coset sigma model in section 4.3 we learned that we can treat the coset constraints either explicitly, by adding them to the Lagrangian, or implicitly. In the implicit formalism we treat the generalised matrix as a generic symmetric matrix of scalar fields, but we have to refrain from using the coset constraints inside the Poisson brackets in this case. In chapter 5 we have successfully made use of the implicit formalism for the treatment of the $\mathrm{E}_{6(6)}$/USp(8) coset constraints in the canonical analysis of the five-dimensional maximal supergravity theory. For the analysis of the $\mathrm{E}_{6(6)}$ ExFT we will likewise use the implicit treatment of the coset constraints in order to simplify the analysis.

One aspect that we are looking at in this investigation of the canonical formulation of ExFT concerns the (physical) role of the section condition (3.72). Before we begin with the canonical analysis we should note that the section condition cannot be interpreted as a canonical constraint itself. The reason for this is that canonical constraints only pose restrictions on the canonical coordinates of phase-space, whereas the section condition is not a condition on any specific function, but instead on the internal coordinate derivatives ∂_M (see section 3.2). The section condition is hence more fundamental than the canonical constraints and it has to be applied ad hoc — as is the case in the Lagrangian formalism. One could try to add the section condition explicitly to the Lagrangian with Lagrange multipliers, but because the section condition applies to any function one would have to add infinitely many terms to the Lagrangian, which in this form does not seem possible.

In this chapter the generalised Lie derivative $\mathbb{L}_\Lambda X$ of any object X always includes a weight term (cf. definition (3.74)), where the generalised diffeomorphism weight $\lambda(X)$ is as stated in table 6.2.

6.2 ADM decomposition of the Lagrangian

In this section we compute the ADM decomposition of all terms in the Lagrangian of the (bosonic) $\mathrm{E}_{6(6)}$ EFT (3.98). The results of this section are later used to compute the canonical momenta in section 6.3 and to carry out the Legendre transformation in section 6.5. The explicit computation of the ADM decomposition of the Lagrangian can furthermore give some intuition and insight into where the various terms originate from and what their role may be. In this section we are working on terms that are part of the Lagrangian and — as we did in chapter 5 — we sometimes discard total

Gen. weight λ	Objects
$-2/3$	$G^{\mu\nu}$, $g^{\mu\nu}$, $\hat{R}$, V_{pot}
$-1/3$	∂_M, $E_\alpha{}^\mu$, $e_a{}^m$
0	∂_μ, $\mathbb{L}_{A_\mu}$, d_{MNK}, M_{MN}, $\mathcal{V}_M{}^{AB}$, $\hat{R}_{\mu\nu}{}^{\alpha\beta}$, N^n
1/3	A_μ^M, $\mathcal{F}_{\mu\nu}^M$, Λ^M, N, N_a, $E_\mu{}^\alpha$, $e_m{}^a$, $\Pi^{stN}(B)$
2/3	$B_{\mu\nu M}$, $\Xi_{\mu M}$, $\mathcal{H}_{\mu\nu\rho M}$, $\Pi^m_M(A)$, $G_{\mu\nu}$, g_{mn}, N_n, $\Pi(e)^m_a$
1	$\mathcal{L}_{\text{ExFT}}$, $\Pi^{MN}(M)$, $\Pi^M{}_{AB}(\mathcal{V})$
4/3	e
5/3	E

Table 6.2: The generalised exceptional diffeomorphism weights of the most important objects in the canonical formulation of $E_{6(6)}$ exceptional field theory.

derivative terms that include the Lagrange multipliers in the derivative, because they are not relevant to the results of this work.

ADM decomposition of the improved Einstein-Hilbert term

In the following we use the ADM decomposition described in section 4.2, but applied only to the external Lorentzian part of the ExFT geometry — the internal extended generalised exceptional geometry is left untouched in this decomposition. We furthermore fix the external Lorentz symmetry partially and use the parametrisation (4.22) to decompose the external vielbein $E_\mu{}^\alpha$ into the lapse function N, the shift vector N^a and the spatial vielbein $e_m{}^a$. These are all functions that also depend on the 27 internal coordinates Y^M. We use the spatial vielbein to flatten or unflatten spatial indices.

The first term in the Lagrangian (3.98) is the improved Einstein-Hilbert term, which is given by (6.1).

$$\mathcal{L}_{\text{EH}} = E\,\hat{R} = E\,R + E\,\mathcal{F}_{\mu\nu}^M\,E^{\alpha\rho}\,\partial_M E_\rho{}^\beta\,E_\alpha{}^\mu\,E_\beta{}^\nu \tag{6.1}$$

We begin by calculating the ADM decomposition of the covariantised Einstein-Hilbert term $E\,R$. This term is covariantised in the sense that the external derivatives ∂_μ have been replaced by the covariant derivatives $\mathcal{D}_\mu$ (3.85). The Ricci scalar is then constructed in terms of the covariantised coefficients of anholonomy, which are defined by (6.2) (cf. section 5.2.1).

$$\Omega_{\alpha\beta\gamma} := 2\,E_{[\alpha}{}^\mu E_{\beta]}{}^\nu\,\mathcal{D}_\mu E_{\nu\gamma}. \tag{6.2}$$

This gauging of the Ricci scalar, which introduces a dependence on the one-forms A_μ^M via the covariant derivative, is what makes the vielbein transform under the generalised diffeomorphisms.

Inserting the ADM decomposition of the vielbein (4.22) into the definition of the covariantised coefficients of anholonomy (6.2) we find the expressions (6.3), (6.4) and (6.5).

$$\Omega_{abc} = 2\,e_{[a}{}^m e_{b]}{}^n\,\mathcal{D}_m e_{nc} \tag{6.3}$$

$$\Omega_{ab0} = 0 \tag{6.4}$$

$$\Omega_{0b0} = -e_b{}^n N^{-1}\,\mathcal{D}_n N \tag{6.5}$$

We saw in section 5.2.1 that in supergravity (and general relativity) only the Ω_{0bc} components (5.4) depend on the time derivative. This is also the case in the ExFT, although the time derivative is now covariantised and the Ω_{0bc} components are given by (6.6).

$$\Omega_{0bc} = \frac{1}{N}\Big(e_b{}^n\,(\mathcal{D}_0 - N^m\mathcal{D}_m)\,e_{nc} - e_b{}^m\,e_{nc}\,\mathcal{D}_m N^n\Big) \tag{6.6}$$

Equation (6.6) can be inverted to give the time derivative of the spatial vielbein as (6.7).

$$\partial_0 e_{kc} = N e_k{}^b\,\Omega_{0bc} + (\mathbb{L}_{A_0} + N^m\mathcal{D}_m)\,e_{kc} + e_{nc}\,\mathcal{D}_k N^n \tag{6.7}$$

With the above decomposition of the coefficients of anholonomy the ADM decomposition of the Einstein-Hilbert term can be written as (6.8), with R_d being the (covariantised) d-dimensional Ricci scalar.

$$E\,R_5 = e\,N\,(\Omega_{0(ab)}\Omega_{0(ab)} - \Omega_{0aa}\Omega_{0bb} + R_4) \tag{6.8}$$

The decomposition (6.8) is of the standard form [54, 205], which we also saw in chapter 5, cf. equation (5.3), but we have to remember that the objects in (6.8) are nonetheless covariantised.

The second term in (6.1) is the Einstein-Hilbert improvement term and we can write its ADM decomposition as (6.9).

$$\begin{aligned} +E\,\mathcal{F}^M_{\mu\nu}\,E^{\alpha\rho}\,\partial_M E_\rho{}^\beta\,E_\alpha{}^\mu\,E_\beta{}^\nu = &+\frac{e}{N}\,\mathcal{F}^M_{tn}\,\partial_M N^n \\ &-\frac{e}{N}\,\mathcal{F}^M_{mn}\,N^m\,\partial_M N^n \\ &+e\,N\,\mathcal{F}^M_{mn}\,e^{ar}\,\partial_M e_r{}^b\,e_a{}^m\,e_b{}^n \end{aligned} \tag{6.9}$$

Due to the time component of the field strength the first term in (6.9) contributes to the canonical momenta of the one-forms. We can note in particular the sign of the second term in (6.9) and we will come back to this point in section 6.5.4. The last term in (6.9) is the spatial Einstein-Hilbert improvement term and together with the spatial Ricci scalar R_4 from (6.8) it will define the spatial improved Ricci scalar $\hat{R}_4$.

ADM decomposition of the Yang-Mills term

Equation (6.10) states the ADM decomposition of the generalised Yang-Mills term.

$$\begin{aligned} -\frac{E}{4}\,M_{MN}\,\mathcal{F}^{\mu\nu M}\,\mathcal{F}^N_{\mu\nu} = &+\frac{e}{2\,N}\,M_{MN}\,\mathcal{F}^M_{ts}\,\mathcal{F}^N_{tn}\,g^{sn} \\ &-\frac{e}{N}\,M_{MN}\,\mathcal{F}^M_{ts}\,\mathcal{F}^N_{mn}\,g^{sn}\,N^m \\ &-\frac{e\,N}{4}\,M_{MN}\,\mathcal{F}^M_{rs}\,\mathcal{F}^N_{mn}\,g^{rm}\,g^{sn} \\ &+\frac{e}{2\,N}\,M_{MN}\,\mathcal{F}^M_{rs}\,\mathcal{F}^N_{mn}\,N^r\,N^m\,g^{sn} \end{aligned} \tag{6.10}$$

The first two terms of (6.10) contribute to the canonical momenta because they are quadratic or linear in the time components of the one form field strength. The quadratic term here is responsible for many terms in the Hamiltonian due to the complicated one-form momenta, which will replace the time derivatives of the one-forms in the Legendre transformation. The third term in (6.10) is the spatial Yang-Mills term. The last term will cancel in the Legendre transformation.

ADM decomposition of the scalar kinetic term

Equation (6.11) states the ADM decomposition of the scalar kinetic term — the structure of these terms will become clearer after the Legendre transformation, but in the last line of (6.11) we can already recognise the spatial scalar kinetic term as the first term.

$$\begin{aligned}
&+\frac{1}{24}\,E\,g^{\mu\nu}\,\mathcal{D}_\mu M_{MN}\,\mathcal{D}_\nu M^{MN} \\
=&-\frac{e}{24\,N}\Big(-\dot{M}_{MN}\,\dot{M}_{RS}\,M^{RM}\,M^{SN}-\dot{M}_{MN}\,\mathbb{L}_{A_t}M^{MN} \\
&\qquad+\mathbb{L}_{A_t}M_{MN}\,\dot{M}_{RS}\,M^{RM}\,M^{SN}+\mathbb{L}_{A_t}M_{MN}\,\mathbb{L}_{A_t}M^{MN}\Big) \\
&+\frac{e}{24\,N}\,N^l\Big(+\dot{M}_{MN}\,\mathcal{D}_l M^{MN}-\dot{M}_{RS}\,M^{RM}\,M^{SN}\,\mathcal{D}_l M_{MN} \\
&\qquad-\mathbb{L}_{A_t}M_{MN}\,\mathcal{D}_l M^{MN}-\mathcal{D}_l M_{MN}\,\mathbb{L}_{A_t}M^{MN}\Big) \\
&+\frac{1}{24}\Big(e\,N\,g^{kl}-\frac{e}{N}\,N^k\,N^l\Big)\,\mathcal{D}_k M_{MN}\,\mathcal{D}_l M^{MN}
\end{aligned} \tag{6.11}$$

ADM decomposition of the topological term

In order to keep the expressions simple we split the topological term (3.135) into its individual constituent terms and compute the space-time splits of these.[1]

First we examine the two-form kinetic term and find that its space-time split is given by (6.12).

$$\begin{aligned}
-\frac{15\kappa}{2}\,\epsilon^{\mu\nu\rho\sigma\tau}\,d^{MNR}\,\partial_R B_{\mu\nu M}\,\mathcal{D}_\rho B_{\sigma\tau N}=&-15\kappa\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{tnM}\,\mathcal{D}_r B_{slN} \\
&+15\kappa\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\mathcal{D}_s B_{tlN} \\
&+\frac{15\kappa}{2}\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\mathbb{L}_{A_t}B_{slN} \\
&-\frac{15\kappa}{2}\,\epsilon^{tnrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\partial_t B_{slN}
\end{aligned} \tag{6.12}$$

The expression (6.12) is the covariantised version of the space-time split (4.55) of the model theory from section 4.4. The only time derivative on the two-forms, in the ExFT Lagrangian, is found in the last term of (6.12). Up to the Stückelberg and topological couplings the dynamics of the two-forms should thus expected to be identical to those of the model in section 4.4.

[1] We use the terms ADM decomposition and space-time split somewhat interchangeably in this chapter.

The space-time decomposition of the two topological terms that couple the one- and two-forms are given by (6.13) and (6.14).

$$\begin{aligned}
&+15\kappa\epsilon^{\mu\nu\rho\sigma\tau}\ d^{MNR}\ d_{NKL}\,\partial_R B_{\mu\nu M}\,A^K_\rho\,\partial_\sigma A^L_\tau \\
=&+30\kappa\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{tnM}\,A^K_r\,\partial_s A^L_l \\
&-15\kappa\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{nrM}\,A^K_s\,\dot{A}^l_l \\
&-15\kappa\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_l\partial_R B_{nrM}\,A^K_s\,A^L_t \\
&+30\kappa\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{nrM}\,A^K_t\,\partial_s A^L_l
\end{aligned} \tag{6.13}$$

We should take note of the time derivative on the one-form in the third line of (6.13) — this leads to the first of three topological contributions to the one-form momenta.

$$\begin{aligned}
&-5\kappa\,\epsilon^{\mu\nu\rho\sigma\tau}\,d^{MNR}\,d_{NKL}\,\partial_R B_{\mu\nu M}\,A^K_\rho\,[A_\sigma, A_\tau]^L_{\mathrm{E}} \\
=&-10\kappa\,\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{tnM}\,A^K_r\,[A_s, A_l]^L_{\mathrm{E}} \\
&-5\kappa\,\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{nrM}\,A^K_t\,[A_s, A_l]^L_{\mathrm{E}} \\
&+10\kappa\,\epsilon^{tnrsl}\,d^{MNR}\,d_{NKL}\,\partial_R B_{nrM}\,A^K_s\,[A_t, A_l]^L_{\mathrm{E}}
\end{aligned} \tag{6.14}$$

The only part of the topological term of the $\mathrm{E}_{6(6)}$ ExFT (3.135) that does not depend on any internal derivatives is the term whose space-time split is given by (6.15).

$$\begin{aligned}
&+\kappa\ \epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A^N_\mu\ \partial_\nu A^M_\rho\ \partial_\sigma A^P_\tau \\
=&+\kappa\ \epsilon^{tnrsl}\,d_{MNP}\ A^N_t\ \partial_n A^M_r\ \partial_s A^P_l \\
&-2\,\kappa\ \epsilon^{tnrsl}\,d_{MNP}\ A^N_n\ \dot{A}^M_r\ \partial_s A^P_l \\
&+2\,\kappa\ \epsilon^{tnrsl}\,d_{MNP}\ A^N_n\ \partial_r A^M_t\ \partial_s A^P_l
\end{aligned} \tag{6.15}$$

Hence (6.15) is the only part of topological term that exists in the trivial solution of the section condition and we can see that it becomes the topological term in the Lagrangian of the $\mathrm{E}_{6(6)}$ invariant formulation of the five-dimensional maximal ungauged supergravity that was analysed in chapter 5. As for the canonical momenta (5.9) of the five-dimensional theory, the time derivative in (6.15) contributes another topological term to the canonical momenta of the one-forms.

The final topological contribution to the one-form momenta comes from the term in the fourth line of the space-time split of the term (6.16).

$$\begin{aligned}
&-\frac{3\kappa}{4}\epsilon^{\mu\nu\rho\sigma\tau} d_{MNP}\ A^N_\mu\ \ [A_\nu, A_\rho]^M_E\ \ \partial_\sigma A^P_\tau \\
=&-\frac{3\kappa}{4}\epsilon^{tnrsl}\,d_{MNP}\Big(+A^N_t\ [A_n, A_r]^M_E\ \partial_s\,A^P_l \\
&\qquad\qquad\qquad -2A^N_n\ [A_t, A_r]^M_E\ \partial_s\,A^P_l \\
&\qquad\qquad\qquad -A^N_n\ [A_r, A_s]^M_E\ \dot{A}^P_l \\
&\qquad\qquad\qquad +A^N_n\ [A_r, A_s]^M_E\ \partial_l\,A^P_t\Big)
\end{aligned} \tag{6.16}$$

The space-time split of the last topological term is given by (6.17).

$$\begin{aligned} &+\frac{3\kappa}{20}\epsilon^{\mu\nu\rho\sigma\tau}d_{MNP}\;A_\mu^N\;\;[A_\nu,A_\rho]_E^M\;\;[A_\sigma,A_\tau]_E^P \\ =&+\frac{3\kappa}{20}\epsilon^{tnrsl}\,d_{MNP}\bigg(+A_t^N\;[A_n,A_r]_E^M\;[A_s,A_l]_E^P \\ &\qquad\qquad\qquad -4\,A_n^N\;[A_t,A_r]_E^M\;[A_s,A_l]_E^P\bigg) \end{aligned} \tag{6.17}$$

The terms (6.13) and (6.17) do not depend on any external derivatives and hence do not contribute to any of the canonical momenta.

ADM decomposition of the scalar potential term

The scalar potential (3.103) decomposes in the ADM split as (6.18) — one should take note here that the potential is already written with the sign that it will take in the Hamiltonian.

$$\begin{aligned} +E\,V_{\text{pot}} =&-\frac{e}{2N}g_{mn}M^{MN}\partial_M N^n\,\partial_N N^m \\ &-\frac{N\,e}{4}M^{MN}\partial_M g^{mn}\,\partial_N g_{mn}-\frac{N}{e}M^{MN}\partial_M e\,\partial_N e \\ &-\frac{N\,e}{24}M^{MN}\partial_M M^{KL}\,\partial_N M_{KL}+\frac{N\,e}{2}M^{MN}\partial_M M^{KL}\,\partial_L M_{NK} \\ &+N\,\partial_M\partial_N M^{MN}\,e+N\,2\,M^{MN}\,\partial_M\partial_N e+N\,2\,\partial_M M^{MN}\,\partial_N e \end{aligned} \tag{6.18}$$

The first term in (6.18) depends on several Lagrange multipliers, this would be problematic in the Hamiltonian formalism, but we will see that this term fortunately cancels in the Legendre transformation. None of the other terms cancel and they will form the scalar potential of the Hamiltonian, which sits inside the Hamilton constraint.

6.3 Canonical momenta

Based on the ADM decomposition of the Lagrangian (3.98) that we have found in section 6.2 we can now go on to compute the canonical momenta of all fields in this section. Moreover we introduce important redefinitions of some canonical coordinates in this section, which are crucial to making the Legendre transformation and the Hamiltonian as simple as possible.

The canonical momenta of the one-forms A_μ^M

In the Lagrangian (3.98) there are no time derivatives on the time components A_t^T of the one-forms, because of the antisymmetry of the field strength and because all of the topological terms are contracted with the Levi-Civita symbol. The canonical momenta (6.19) of the time components A_t^T hence vanish.

$$\Pi_T^t(A) = 0 \tag{6.19}$$

$$\begin{aligned}\Pi_T^l(A) =& \frac{e}{N} g^{ln}\ M_{TN}\ \left(\mathcal{F}_{tn}^N + N^k \mathcal{F}_{nk}^N\right) \\ &- \frac{3\kappa}{4}\epsilon^{lmnr} d_{MNT} A_m^N\ [A_n, A_r]_E^M \\ &+ 2\kappa\ \epsilon^{lmnr} d_{MNT}\ A_m^M\ \partial_n A_r^N \\ &+ 15\kappa\ \epsilon^{lmnr} d^{MNR} d_{NKT}\ \partial_R B_{mnM}\ A_r^K \\ &+ \frac{e}{N}\partial_T N^l\end{aligned} \tag{6.20}$$

Equation (6.20) states the canonical momenta of the spatial components of the one-forms A_l^T. The first line in (6.20) comes from the covariantised generalised Yang-Mills term. The topological contributions to (6.20) are the next three terms. We can compare these terms in (6.20) to the canonical momenta (5.9) of the five-dimensional theory, although there are of course more topological contributions in ExFT. The final $\frac{e}{N}\partial_T N^l$ term in (6.20) is the contribution from the Einstein-Hilbert improvement term (6.9) and it does not have any analogue in the five-dimensional supergravity theory.

One of the main lessons that we can learn from the canonical analysis in chapter 5 is the treatment of the topological contributions to the canonical momenta. For the five-dimensional supergravity we defined modified momenta-like variables (5.16) by subtracting all topological contributions from the canonical momenta $\Pi_T^l(A)$. The resulting modified variables then took the form (5.17), which is just the terms that one would expect in Yang-Mills theory coupled to gravity. We found that this greatly simplified the Hamiltonian and that the gauge transformations of these modified momenta took a nice form. In the ExFT we can define the modified momenta-like variables $\mathcal{P}_T^l(A)$ by analogy as (6.21), where we subtract the three topological contributions.

$$\begin{aligned}\mathcal{P}_T^l(A) :=& + \Pi_T^l(A) \\ &+ \frac{3\kappa}{4}\epsilon^{lmnr} d_{MNT} A_m^N\ [A_n, A_r]_E^M \\ &- 2\kappa\ \epsilon^{lmnr} d_{MNT}\ A_m^M\ \partial_n A_r^N \\ &- 15\kappa\ \epsilon^{lmnr} d^{MNR} d_{NKT}\ \partial_R B_{mnM}\ A_r^K\end{aligned} \tag{6.21}$$

$$= + \frac{e}{N} g^{ln}\ M_{TN}\ \left(\mathcal{F}_{tn}^N + N^k \mathcal{F}_{nk}^N\right) + \frac{e}{N}\partial_T N^l \tag{6.22}$$

The $\mathcal{P}_T^l(A)$ can hence be written explicitly as (6.22). In order to arrive at the most concise Hamiltonian it turns out that the $\frac{e}{N}\partial_T N^l$ term should not be subtracted from the momenta.

We will see that the modified momenta $\mathcal{P}_T^l(A)$ do indeed greatly simplify the Hamiltonian — without them there are many topological terms scattered in all of the secondary constraints. The Legendre transformation with respect to the canonical momenta $\Pi_T^l(A)$ is very messy because there is a large number of topological terms involved. But as we will see in section 6.5, we can write the Legendre transformation with respect to $\mathcal{P}^l_T(A)$ in a comparatively compact form.

The definition (6.21) is not a canonical transformation. As we have already seen in chapter 5, redefinitions of this form lead to variables that do not Poisson-commute with themselves $\{\mathcal{P}^n_N(A), \mathcal{P}^m_M(A)\} \neq 0$. We saw that the explicit Poisson brackets of these variables can be rather complicated, see equation (5.18), and because of the additional topological contributions in the momenta the analogous relation is much worse in ExFT. The Poisson-noncommutativity of $\mathcal{P}^n_N(A)$ represents one of the greatest difficulties in the canonical analysis of the $E_{6(6)}$ ExFT. This is further exacerbated by the very complicated Hamiltonian topological term (which is analogous to the rather simple F^2 term in the Gauß constraint (5.34) of the five-dimensional theory). These difficulties are not avoided if we use the canonical momenta $\Pi^l_T(A)$ instead, because in this case the Hamiltonian itself is already much more complicated.

As in chapter 5, we can use the topological coefficient κ to break up the computations into orders of κ. By definition the modified momenta agree with the canonical momenta $\Pi^n_N(A) = \mathcal{P}^n_N(A)$ at $\kappa = 0$. The main structure of most computations is already apparent at $\kappa = 0$. An exception to this are all computations that primarily concern the two-forms, whose entire dynamics is topological. In some cases the full computation may turn out to be exceedingly difficult and we only give the results at $\kappa = 0$.

We should emphasise that the case $\kappa = 0$ is only used as a *computational tool*, which allows us to remove one of the main difficulties — i.e. the topological contributions — from the calculations. It should not be expected that the limit $\kappa = 0$ leads to any physically meaningful theory when the section condition is solved.

The canonical momenta of the two-forms $B_{\mu\nu M}$

The only term in the Lagrangian that has a time derivative on the two-forms is the topological kinetic term (6.12). Therefore the canonical momenta (6.23) and (6.24) are identical to those of the model two-form theory that we examined in section 4.4.1.

$$\Pi^{tl\,N}(B) = 0 \tag{6.23}$$

$$\Pi^{sl\,N}(B) = -\,15\kappa\;\epsilon^{mnsl} d^{MNR}\,\partial_R B_{mnM} \tag{6.24}$$

The canonical momenta of the scalar fields M_{MN}

The calculation of the canonical momenta of the scalar fields involves a minor technical subtlety that we glossed over in chapter 5, but it may be beneficial to be precise and to briefly explain this point because it concerns the Legendre transformation in section 6.5.

The issue concerns the scaling of the diagonal components of the scalar matrix M_{MN}. Both $\frac{\partial \dot{M}_{11}}{\partial \dot{M}_{11}} = 1$ and $\frac{\partial \dot{M}_{12}}{\partial \dot{M}_{12}} = 1$ should be true (and similarly for the other components), but this is equivalent to (6.25), which is rather unappealing due to the Kronecker delta term.

$$\frac{\partial \dot{M}_{QP}}{\partial \dot{M}_{MN}} = \delta^M_Q \delta^N_P + \delta^M_P \delta^N_Q - (\delta^{\text{Kronecker}}_{MN})\delta^M_Q \delta^N_P \tag{6.25}$$

With (6.25) the canonical momenta of the scalar fields are computed to be (6.26) (the indices R, S are not summed over in this expression).

$$\Pi^{RS}(M) = \frac{e}{12\,N}(2-\delta^{\text{Kronecker}}_{RS})\left[+\dot{M}_{QP}\;M^{QR}M^{PS}+N^n\;\mathcal{D}_nM^{RS}+\mathbb{L}_{A_t}M^{RS}\right] \quad (6.26)$$

In order to get rid of the Kronecker delta term in (6.26) we can introduce the rescaling (6.27) of the diagonal components of the canonical momenta by a factor of 2.

$$\tilde{\Pi}^{RS}(M) := \begin{cases} 2\cdot\Pi^{RS}(M), & \text{if} \quad R=S \\ \quad \Pi^{RS}(M), & \text{if} \quad R\neq S \end{cases} \quad (6.27)$$

Explicitly the rescaled momenta $\tilde{\Pi}^{RS}(M)$ are then given by (6.28).

$$\tilde{\Pi}^{RS}(M) = \frac{e}{6\,N}\;\left[+\dot{M}_{QP}\;M^{QR}M^{PS}+N^n\;\mathcal{D}_nM^{RS}+\mathbb{L}_{A_0}M^{RS}\right] \quad (6.28)$$

The distinction between the rescaled and original canonical momenta is only relevant during the Legendre transformation in section 6.5, where it is needed to find the correct coefficients of the scalar terms. After section 6.5 the distinction is irrelevant and we simply write $\Pi^{RS}(M) = \tilde{\Pi}^{RS}(M)$.

The canonical momenta of the external metric components N, N_a, ${e_m}^a$

We have seen that the Ricci scalar only contains a time derivative in the Ω_{0bc} components (6.6) of the coefficients of anholonomy, which is on the spatial vielbein. The canonical momenta of the lapse function (6.29) and the shift vector (6.30) are vanishing.

$$\Pi(N) = 0 \quad (6.29)$$
$$\Pi^a(N_a) = 0 \quad (6.30)$$
$$\Pi^m_a(e) = 2\,e\,e^m_b\,(\Omega_{0(ab)} - \delta_{ab}\,\Omega_{0cc}) \quad (6.31)$$

The canonical momenta of the spatial vielbein are given by (6.31), which is of the same form as (5.8), but written in terms of the covariantised coefficients of anholonomy. As in chapter 5 we define the contractions $\Pi_{ab}(e)$ (6.32) and $\Pi(e)$ (6.33) of the canonical momenta of the spatial vielbein.

$$\Pi_{ab}(e) := +\,e_{m(a}\,{\Pi^m}_{b)}(e) \quad (6.32)$$
$$\Pi(e) := +\,{e_m}^a\,{\Pi^m}_a(e) \quad (6.33)$$

6.4 Primary constraints

In section 6.3 we have computed all of the canonical constraints, which we can now use to identify the complete set of primary canonical constraints.

We have found that the canonical momenta of the lapse function (6.34), of the shift vector (6.35) and of the time components of the differential forms (6.36) and (6.37) vanish.

$$\Pi(N) = 0 \tag{6.34}$$
$$\Pi^a(N_a) = 0 \tag{6.35}$$
$$\Pi_M(A_t^M) = 0 \tag{6.36}$$

These are all primary constraints of shift type, analogous to those we have already seen in chapter 5, which only generate shift transformations on the canonically conjugate fields. We will see that all of the fields conjugate to these momenta are Lagrange multipliers and that the consistency of each of the shift type primary constraints generates a secondary constraint.

In analogy to the model of section 4.4 we find the primary constraints $\mathcal{H}_{\mathrm{P1}}$ (6.37) and $\mathcal{H}_{\mathrm{P2}}$ (6.38) coming from the two-form momenta.

$$(\mathcal{H}_{\mathrm{P1}})^{mM} := \Pi^{tm\,M}(B) = 0 \tag{6.37}$$
$$(\mathcal{H}_{\mathrm{P2}})^{slN} := \left(\Pi^{sl\,N}(B) + 15\,\kappa\,\epsilon^{tmnsl} d^{MNR}\,\partial_R B_{mnM}\right) = 0 \tag{6.38}$$

As was discussed in section 4.4 the constraints (6.38) exist because the topological kinetic term of the two-forms is linear in the time derivative, which has numerous important consequences, as we will see.

The Lorentz constraints of ExFT (6.39) are analogous to those of five-dimensional supergravity (or pure general relativity) and have already be discussed in chapter 5.

$$L_{ab} := e_{m[a}\Pi(e)^m{}_{b]} = 0 \tag{6.39}$$

In total we have found $1 + 4 + 27 + 108 + 162 + 6 = 308$ primary constraints.

6.5 Legendre transformation

In chapter 5 we glossed over the Legendre transformation of the five-dimensional theory as it was quite straightforward. For the $\mathrm{E}_{6(6)}$ exceptional field theory the Legendre transformation is significantly more complicated and we want to discuss the calculation in this section. First we explain how the calculation can be broken up into the transformations of each sector of the theory. Then we carry out the partial calculations using the results of section 6.2.

Formally we can write the Legendre transformation of the bosonic Lagrangian $\mathcal{L}_{\text{ExFT}}$ as (6.40).

$$\begin{aligned}
\mathcal{H}_{\text{ExFT}} = \quad & + \dot{N} \cdot \Pi(N) + \sum_{a=1,\ldots,4} \dot{N}_a \cdot \Pi^a(N_a) + \sum_{\substack{a=1,\ldots,4 \\ m=1,\ldots,4}} \dot{e}_{ma} \cdot \Pi^{ma}(e_{ma}) \\
& + \sum_{N=1,\ldots,27} \dot{A}_t^N \cdot \Pi_N(A_t^N) + \sum_{\substack{N=1,\ldots,27 \\ n=1,\ldots,4}} \dot{A}_n^N \cdot \Pi_N^n(A_n^N) \\
& + \sum_{\substack{N=1,\ldots,27 \\ n=1,\ldots,4}} \dot{B}_{tnN} \cdot \Pi^{nN}(B_{tnN}) + \frac{1}{2} \sum_{\substack{N=1,\ldots,27 \\ m,n=1,\ldots,4}} \dot{B}_{mnN} \cdot \Pi^{mnN}(B_{mnN}) \\
& + \sum_{\substack{R,S=1,\ldots,27 \\ R\leq S}} \dot{M}_{RS} \cdot \Pi^{RS}(M_{RS}) - \mathcal{L}_{\text{ExFT}}
\end{aligned} \tag{6.40}$$

In order to avoid overcounting we explicitly write out the sums in (6.40) and only sum over the independent components. For the spatial two-form components we can insert a factor of 1/2 and sum over all components. Due to the use of the implicit treatment of the coset constraints (cf. section 4.3) we can treat the scalar fields as a generic symmetric matrix. We cannot simply sum over all components of the scalar fields because that would give the wrong coefficient for the diagonal terms. Using the rescaled scalar momenta (6.27) we can write the transformation term as (6.41) where we can now sum over all components.

$$\sum_{R\leq S} \Pi^{RS}(M)\dot{M}_{RS} = \frac{1}{2} \sum_{R,S=1,\ldots,27} \tilde{\Pi}^{RS}(M)\dot{M}_{RS} \tag{6.41}$$

Inserting the Lagrangian (3.98) we can write the Legendre transformation as (6.42), where the lines are already separated into the various sectors of the theory.

$$\begin{aligned}
\mathcal{H}_{\text{ExFT}} = \quad & + \sum_{\substack{a=1,\ldots,4 \\ m=1,\ldots,4}} \dot{e}_{ma} \cdot \Pi^{ma}(e_{ma}) - E\, R_5 \\
& + \sum_{\substack{N=1,\ldots,27 \\ n=1,\ldots,4}} \dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \mathcal{L}_{\text{YM}} - {}^{(5)}E\, \mathcal{F}^M_{\alpha\beta} E^{\alpha\rho} \partial_M E^\beta_\rho \\
& + \frac{1}{2} \sum_{\substack{N=1,\ldots,27 \\ m,n=1,\ldots,4}} \dot{B}_{mnN} \cdot \Pi^{mnN}(B_{mnN}) - \mathcal{L}_{\text{top}} \\
& + \frac{1}{2} \sum_{R,S=1,\ldots,27} \dot{M}_{RS} \cdot \tilde{\Pi}^{RS}(M_{RS}) - \mathcal{L}_{\text{sc}} - \mathcal{L}_{\text{pot}} \\
& + \dot{N} \cdot \Pi(N) + \sum_{a=1,\ldots,4} \dot{N}_a \cdot \Pi^a(N_a) \\
& + \sum_{N=1,\ldots,27} \dot{A}_t^N \cdot \Pi_N(A_t^N) + \sum_{\substack{N=1,\ldots,27 \\ n=1,\ldots,4}} \dot{B}_{tnN} \cdot \Pi^{nN}(B_{tnN})
\end{aligned} \tag{6.42}$$

Next we compute the Legendre transformation for each sector separately in the following sections. Time derivatives of the one-forms appear in the generalised Yang-Mills term, the Einstein-Hilbert improvement term and in the topological term, which makes the one-form sector the most difficult and we look at it in more detail.

6.5.1 Legendre transformation of the Einstein-Hilbert term

Because the Einstein-Hilbert term does not contain any time derivatives of the lapse function or shift vector the Legendre transformation terms of these fields stay in the Hamiltonian as primary constraint terms. The Legendre transformation of the Einstein-Hilbert term (without the improvement term) with respect to the spatial vielbein is given by (6.43).

$$\begin{aligned}\Pi^m_a(e)\,\dot{e}_{ma} - E\,R_5 = &+ N\cdot\left(\frac{1}{4e}\,\Pi_{ab}(e)\,\Pi_{ab}(e) - \frac{1}{12e}\Pi^2(e) - e\,R_4\right)\\ &+ N^n\cdot\left(2\,\Pi^m{}_a(e)\,\mathcal{D}_{[n}e_{m]a} - e_{na}\,\mathcal{D}_m\Pi^m{}_a(e)\right)\\ &+ A_t^K\cdot\left(\Pi^m{}_a(e)\,\partial_K e_{ma} - \frac{1}{3}\partial_K\Pi(e)\right)\end{aligned} \tag{6.43}$$

In (6.43) we have already factored out the Lagrange multipliers, which makes the contributions to the secondary constraints apparent. We can recognise the covariantised version of the pure general relativity Hamiltonian [13, 203, 205] in the first two lines of (6.43), which we can also compare this to the Hamiltonian (5.19) from chapter 5. Due to the one-form connection terms in the Ricci scalar we find additional A_t^K terms, which are needed to generate the generalised diffeomorphism transformations of the vielbein and of its canonical momenta.

6.5.2 Legendre transformation of the scalar kinetic term

The scalar potential changes sign in the Legendre transformation, but only the first term in (6.18) cancels, which is discussed in section 6.6. The Legendre transformation of the scalar kinetic term is given by (6.44).

$$\begin{aligned}&\frac{1}{2}\sum_{R,S=1,\dots,27}\dot{M}_{RS}\cdot\tilde{\Pi}^{RS}(M_{RS}) - \mathcal{L}_{\rm sc}\\ &= + N\cdot\left(\frac{3}{2\,e}\,\tilde{\Pi}^{MN}(M)\,\tilde{\Pi}^{RS}(M)\,M_{MR}\,M_{NS} - \frac{e}{24}\,g^{kl}\,\mathcal{D}_kM_{MN}\,\mathcal{D}_lM^{MN}\right)\\ &\quad + N^n\cdot\left(\frac{1}{2}\,\tilde{\Pi}^{MN}(M)\,\mathcal{D}_nM_{MN}\right)\\ &\quad + A_t^K\cdot\left(\frac{1}{2}\,\tilde{\Pi}^{MN}(M)\,\partial_KM_{MN} - 6\,\mathbb{P}^R{}_L{}^S{}_K\,\partial_S\left(\tilde{\Pi}^{LN}(M)\,M_{RN}\right)\right)\end{aligned} \tag{6.44}$$

The first two lines of (6.44) are the contributions to the Hamilton and external diffeomorphism constraints, which are the covariantised version of the terms found in the Hamiltonian (5.19) from chapter 5. Due to the covariantised derivatives we find the additional A_t^K terms, which generate the generalised diffeomorphism transformations on the scalar canonical variables. The adjoint projector $\mathbb{P}^R{}_L{}^S{}_K$ appears in (6.44) because we factorised out the Lagrange multiplier A_t^K from the generalised Lie derivative terms in the ADM decomposition (6.11). The adjoint projector terms only appear for the fields that carry $E_{6(6)}$ indices and hence we do not see such terms in the vielbein sector.

6.5.3 Legendre transformation of the two-form kinetic term

The Legendre transformation of the two-form kinetic term $\mathcal{L}_{\mathrm{BK}}$, defined by (6.45), can be separated from the Legendre transformation of the remaining terms.

$$\mathcal{L}_{\mathrm{BK}} := -\frac{15\kappa}{2}\epsilon^{\mu\nu\rho\sigma\tau}d^{MNR}\,\partial_R B_{\mu\nu M}\,\mathcal{D}_\rho B_{\sigma\tau N} \tag{6.45}$$

The remainder of the terms coming from the topological term $\mathcal{L}_{\mathrm{top}} - \mathcal{L}_{\mathrm{BK}}$ are transformed together with the other one-form terms in section 6.5.4. The term (6.45) is the covariantised version of the model Lagrangian (4.48) from section 4.4. The Legendre transformation of $\mathcal{L}_{\mathrm{BK}}$ is given by (6.46).

$$\begin{aligned}&\frac{1}{2}\,\dot{B}_{mnN}\cdot\Pi^{mnN}(B_{mnN}) - \mathcal{L}_{\mathrm{BK}} \qquad\qquad (6.46)\\ &= -\,\frac{15\kappa}{2}\,\epsilon^{nrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\mathbb{L}_{A_t}B_{slN} - 30\kappa\,\epsilon^{nrsl}\,d^{MNR}\,B_{tnM}\,\mathcal{D}_r\partial_R B_{slN}\end{aligned}$$

In (6.46) we can see the covariantisation of the $\mathcal{H}_{S1}$ constraints from section 4.4. Due to the covariant derivatives we find another A_t dependent term, which does not have an analogue in the model of section 4.4. For the model theory we were able to express the B_{tnM} term as (4.61) by making use of the naive two-form field strength. Here we are indeed allowed to freely exchange the internal derivative with the covariant derivative because both the internal derivative and the two-forms are contracted with the d-symbol as $d^{MNR}\partial_R B_{slN}$ and we can apply the identity (2.36) of reference [25]. But we cannot yet rewrite the B_{tnM} term in terms of the covariantised field strength $\mathcal{H}_{lmnL}$, because we are missing the one-form terms, which will however appear in the remainder of the Legendre transformation.

We should now examine the A_t dependent term in (6.46) more closely. This term again exists due to the covariant derivative and hence we expect that it is related to the generalised diffeomorphism transformations. Because the two-form kinetic term is linear in the time derivative there are no canonical two-form momenta in (6.46). By inserting the definition of the primary constraints (6.38) we can express this term as (6.47).

$$\begin{aligned}&-\,\frac{15\kappa}{2}\,\epsilon^{nrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\mathbb{L}_{A_t}B_{slN}\\ &= +\,\frac{1}{2}\left(\Pi^{slN}(B) - (\mathcal{H}_{P2})^{slN}\right)\mathbb{L}_{A_t}B_{slN} \qquad\qquad (6.47)\\ &= A_t^M\cdot\Bigg(+\,\frac{1}{2}\,\Pi^{lnN}(B)\,\partial_M B_{lnN} - 3\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left(\Pi^{lnK}(B)\,B_{lnR}\right)\\ &\quad -\,\frac{1}{2}\,(\mathcal{H}_{\mathrm{P2}})^{lnN}\,\partial_M B_{lnN} + 3\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left((\mathcal{H}_{\mathrm{P2}})^{lnK}\,B_{lnR}\right)\\ &\quad -\,\frac{1}{3}\,\partial_M\left(B_{mnN}\Pi^{mnN}(B)\right) + \frac{1}{3}\,\partial_M\left(B_{mnN}(\mathcal{H}_{\mathrm{P2}})^{mnN}\right)\Bigg) \qquad\qquad (6.48)\end{aligned}$$

Written in this form the first term in (6.47) seems to be what we are looking for to generate generalised diffeomorphism transformations on the two-form canonical variables. We are however not allowed to go to the primary constraint surface and set $\mathcal{H}_{P2} = 0$ because this is only allowed after Dirac brackets have been constructed — this point is further discussed in section 6.9.2. Factorising out the Lagrange multipliers we arrive at (6.48), which again makes adjoint projector $\mathbb{P}^R{}_K{}^S{}_M$ visible.

Without using the primary constraints we can expand the generalised Lie derivative directly in (6.46) and factor out the Lagrange multiplier A_t^M to find (6.49).

$$\begin{aligned} & -\frac{15\kappa}{2}\,\epsilon^{nrsl}\,d^{MNR}\,\partial_R B_{nrM}\,\mathbb{L}_{A_t} B_{slN} \\ & = A_t^M \cdot \Big(-75\,\kappa\,\epsilon^{nrsl}\,d^{QNR}\,d^{LST}\,d_{NMT}\,\partial_R B_{nrQ}\,\partial_S B_{slL} \Big) \end{aligned} \tag{6.49}$$

To arrive at the result (6.49) one needs to make use of the section condition (3.72). If we were to use (6.49) in the Hamiltonian it would be the only instance of the section condition being used in the calculation of the ExFT Hamiltonian. This can be avoided by using (6.47) instead. The form (6.48) has the further advantage that it makes the generalised Lie derivative apparent. Depending on the situation we will have to make use of both (6.48) and (6.49) in the following.

6.5.4 Legendre transformation of the Yang-Mills, Einstein-Hilbert improvement and topological terms

The terms whose Legendre transformation we have not yet computed are the generalised Yang-Mills term, the Einstein-Hilbert improvement term and the $(\mathcal{L}_{\text{top}} - \mathcal{L}_{\text{BK}})$ part of the topological term. In this section we compute their Legendre transformation (6.50) with respect to the time derivative of the one-forms.

$$\dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \mathcal{L}_{\text{YM}} - E\,\mathcal{F}^M_{\alpha\beta} E^{\alpha\rho}\partial_M E_\rho^\beta - (\mathcal{L}_{\text{top}} - \mathcal{L}_{\text{BK}}) \tag{6.50}$$

The Legendre transformation of the one-form sector is the most complicated part of the computation of the canonical Hamiltonian. We define the expression Υ_s^M by (6.51), which simplifies the calculation somewhat, but we only need it in this section.

$$\Upsilon_s^M := \mathcal{F}_{ts}^M - \dot{A}_s^M \tag{6.51}$$

$$= -\partial_s A_t^M - [A_t, A_s]_{\text{E}}^M + 10\,d^{MNK}\,\partial_N B_{tsK} \tag{6.52}$$

It is however the use of the modified one-form momenta $\mathcal{P}_T^l(A)$ that really bring the following calculation in a relatively compact form.

First we deal with the terms that depend on the time derivative, i.e. terms with a $\dot{A}_n^N$ or $\mathcal{F}_{tl}^M$. In this calculation the '. . . ' always stand for the same terms which do not depend on any time derivatives — once we have dealt with the time derivatives the terms in the '. . . ' will be written explicitly. Starting from the Legendre transformation (6.53) we insert the ADM decomposition of the terms, we then use (6.51) to rewrite the time components of the field strength. After some rearrangements we arrive at

(6.54).

$$
\begin{aligned}
&\dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \mathcal{L}_{\text{YM}} - E\, \mathcal{F}_{\alpha\beta}^M E^{\alpha\rho} \partial_M E_\rho^\beta - (\mathcal{L}_{\text{top}} - \mathcal{L}_{\text{BK}}) \qquad (6.53)\\
&= \dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \frac{e}{2\,N}\, M_{MN}\, \mathcal{F}_{ts}^M\, \mathcal{F}_{tn}^N\, g^{sn} + \frac{e}{N}\, M_{MN}\, \mathcal{F}_{ts}^M\, \mathcal{F}_{mn}^N\, g^{sn}\, N^m\\
&\quad - \frac{e}{N}\, \mathcal{F}_{tn}^M\, \partial_M N^n + 15\kappa\, \epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{nrM}\, A_s^K\, \dot{A}_l^l\\
&\quad + 2\,\kappa\, \epsilon^{tnrsl}\, d_{MNP}\;\; A_n^N\;\; \partial_t A_r^M\;\; \partial_s A_l^P - \frac{3\kappa}{4} \epsilon^{tnrsl}\, d_{MNP} A_n^N\;\; [A_r, A_s]_E^M\;\; \dot{A}_l^P + \ldots\\
&= \dot{A}_n^N \cdot \Pi_N^n(A_n^N) \qquad (6.54)\\
&\quad - \frac{e}{N}\, M_{MN}\, \mathcal{F}_{ts}^M\, \dot{A}_n^N\, g^{sn} + \frac{e}{2\,N}\, M_{MN}\, (\dot{A}_s^M + \Upsilon_s^M)\, (\dot{A}_n^N - \Upsilon_n^N)\, g^{sn}\\
&\quad + \frac{e}{N}\, M_{MN}\, (\dot{A}_s^M + \Upsilon_s^M)\, \mathcal{F}_{mn}^N\, g^{sn}\, N^m\\
&\quad - \frac{e}{N}\, (\dot{A}_n^M + \Upsilon_n^M)\, \partial_M N^n + 15\kappa\, \epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{nrM}\, A_s^K\, \dot{A}_l^l\\
&\quad + 2\,\kappa\, \epsilon^{tnrsl}\, d_{MNP}\;\; A_n^N\;\; \partial_t A_r^M\;\; \partial_s A_l^P - \frac{3\kappa}{4} \epsilon^{tnrsl}\, d_{MNP} A_n^N\;\; [A_r, A_s]_E^M\;\; \dot{A}_l^P + \ldots
\end{aligned}
$$

To get rid of the Legendre transformation term $\dot{A}_n^N \cdot \Pi_N^n(A_n^N)$ we need to compare (6.54) to the canonical momenta (6.20) — we find that if we insert the explicit expression for $\Pi_N^n(A_n^N)$ most of the time derivative terms cancel. We arrive at (6.55) and time derivatives are now only found in the first term.

$$
\begin{aligned}
&\dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \mathcal{L}_{\text{YM}} - E\, \mathcal{F}_{\alpha\beta}^M E^{\alpha\rho} \partial_M E_\rho^\beta - (\mathcal{L}_{\text{top}} - \mathcal{L}_{\text{BK}}) \qquad (6.55)\\
&= + \frac{e}{2\,N}\, M_{MN}\, (\dot{A}_s^M + \Upsilon_s^M)\, (\dot{A}_n^N - \Upsilon_n^N)\, g^{sn}\\
&\quad + \frac{e}{N}\, M_{MN}\, \Upsilon_s^M\, \mathcal{F}_{mn}^N\, g^{sn}\, N^m - \frac{e}{N}\, \Upsilon_n^M\, \partial_M N^n + \ldots
\end{aligned}
$$

The remaining time derivatives in (6.55) can only be replaced by inverting the canonical momenta. If we were to use the canonical momenta $\Pi_T^l(A)$ here, we would generate many topological terms. If we use the modified momenta $\mathcal{P}_T^l(A)$ instead however, we only generate the terms that are needed. Using (6.51) we can write $\mathcal{P}_T^l(A)$ as (6.56), which we can invert to express $\dot{A}_n^N$ as (6.57).

$$
\mathcal{P}_T^l(A) = \frac{e}{N} g^{ln}\, M_{TN}\, \left(\dot{A}_n^N + \Upsilon_n^N + N^k \mathcal{F}_{nk}^N \right) + \frac{e}{N} \partial_T N^l \qquad (6.56)
$$

$$
\Rightarrow \dot{A}_n^N = \frac{N}{e}\, g_{ln}\, M^{TN} \left(\mathcal{P}_T^l(A) - \frac{e}{N} \partial_T N^l \right) - \Upsilon_n^N - N^k \mathcal{F}_{nk}^N \qquad (6.57)
$$

With (6.57) we can replace the time derivatives in (6.55) with the canonical momenta to find (6.58).

$$
\begin{aligned}
&\dot{A}_n^N \cdot \Pi_N^n(A_n^N) - \mathcal{L}_{\text{YM}} - E\, \mathcal{F}_{\alpha\beta}^M E^{\alpha\rho} \partial_M E_\rho^\beta - (\mathcal{L}_{\text{top}} - \mathcal{L}_{\text{BK}})\\
&= + \frac{N}{2\,e}\, g_{lm}\, M^{KL}\, \mathcal{P}_L^l(A)\, \mathcal{P}_K^m(A) + \frac{e}{2\,N}\, g_{lm}\, M^{KL} \partial_L N^l\, \partial_K N^m\\
&\quad - g_{lm}\, M^{KL} \mathcal{P}_L^l(A)\, \partial_K N^m - \mathcal{P}_N^n(A)\, \Upsilon_n^N + \frac{e}{N}\, N^n\, \mathcal{F}_{mn}^M\, \partial_M N^m\\
&\quad + N^n\, \mathcal{F}_{nl}^M\, \mathcal{P}_M^l(A) + \frac{e}{2\,N}\, g^{mn}\, M_{MN}\, N^k\, N^l\, \mathcal{F}_{mk}^M\, \mathcal{F}_{nl}^N + \ldots \qquad (6.58)
\end{aligned}
$$

It is useful to examine these terms more closely. It is the interplay of the Einstein-Hilbert improvement term and the generalised Yang-Mills term that generates the

term $+\frac{e}{2N}\, g_{lm}\, M^{KL}\partial_L N^l\, \partial_K N^m$. The Einstein-Hilbert improvement term contributes the $\frac{e}{N}\partial_N N^n$ term to $\mathcal{P}^n_N(A)$, which when inserted via (6.57) in the $\dot{A}^2$ Yang-Mills term generates the term in question. This term cancels against the first term in the scalar potential (6.18). This is the only cancellation with the potential and we can furthermore note that the sign of the Einstein-Hilbert improvement term is irrelevant in this cancellation, because of $\dot{A}^2$.
The same interplay of terms generates the term $-g_{lm}\, M^{KL}\mathcal{P}^l_L(A)\, \partial_K N^m$, which is discussed in detail in section 6.9.3, where we discuss the external diffeomorphism transformations.

By inserting the definition of Υ^N_n into the $-\mathcal{P}^n_N(A)\, \Upsilon^N_n$ term we find (6.59).

$$-\mathcal{P}^n_N(A)\, \Upsilon^N_n = \partial_n A^M_t\, \mathcal{P}^n_M(A) + \mathcal{P}^n_M(A)\ [A_t, A_n]^M_{\mathrm{E}} - 10\, d^{MKL}\, \partial_K B_{tnL}\, \mathcal{P}^n_M(A) \quad (6.59)$$

The first two terms of (6.59) can be rewritten as (6.60), which is the covariantised version of the $\mathrm{U}(1)^{27}$ Gauß constraint from chapter 5 plus an additional momentum term that has no analogue in five dimensions.

$$\begin{aligned}&\partial_n A^M_t\, \mathcal{P}^n_M(A) + \mathcal{P}^n_M(A)\ [A_t, A_n]^M_{\mathrm{E}}\\ =&A^M_t \cdot \big(-\mathcal{D}_n\mathcal{P}^n_M(A) - 5\, d^{KLR}\, d_{MRS}\, A^S_n\, \partial_L\mathcal{P}^n_K(A)\big)\end{aligned} \quad (6.60)$$

It is the E-bracket in the field strength of the one-forms $\mathcal{F}^M_{mn}$ that leads to the additional momentum term $A^M_t\, \big(-5\, d^{KLR}\, d_{MRS}\, A^S_n\, \partial_L\mathcal{P}^n_K(A)\big)$ in (6.60). The E-bracket is similarly responsible for the covariantisation of the term $-\mathcal{D}_n\mathcal{P}^n_M(A)$. We will see that the additional momentum term in (6.60) is related to the tensor gauge transformations of the two-forms.

Writing out the many terms that we have so far hidden inside the '. . . ' in (6.58), we find that there are some cancellations and we arrive at equation (6.61).

$$
\begin{aligned}
&\dot{A}_n^N \cdot \Pi^n_N(A_n^N) - \mathcal{L}_{\mathrm{YM}} - E\,\mathcal{F}^M_{\alpha\beta} E^{\alpha\rho}\partial_M E^\beta_\rho - (\mathcal{L}_{\mathrm{top}} - \mathcal{L}_{\mathrm{BK}}) \\
= &+ \frac{N}{2\,e}\, g_{lm}\, M^{KL}\, \mathcal{P}^l_L(A)\, \mathcal{P}^m_K(A) + \frac{e}{2\,N}\, g_{lm}\, M^{KL} \partial_L N^l\, \partial_K N^m \\
&- g_{lm}\, M^{KL} \mathcal{P}^l_L(A)\, \partial_K N^m - 10\, d^{MKL}\, \partial_K B_{tnL}\, \mathcal{P}^n_M(A) + N^n\, \mathcal{F}^M_{nl}\, \mathcal{P}^l_M(A) \\
&- A_t^M \mathcal{D}_n \mathcal{P}^n_M(A) - 5\, A_t^M\, d^{KLR}\, d_{MRS}\, A_n^S\, \partial_L \mathcal{P}^n_K(A) \\
&- e\, N\, \mathcal{F}^M_{mn}\, e^{ar}\, \partial_M e_r{}^b\, e_a{}^m\, e_b{}^n + \frac{e\,N}{4}\, M_{MN}\, \mathcal{F}^M_{rs}\, \mathcal{F}^N_{mn}\, g^{rm}\, g^{sn} \\
&- 30\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{tnM}\, A_r^K\, \partial_s A_l^L \\
&+ 15\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_l \partial_R B_{nrM}\, A_s^K\, A_t^L \\
&- 30\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{nrM}\, A_t^K\, \partial_s A_l^L \\
&+ 10\kappa\, \epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{tnM}\, A_r^K\, [A_s, A_l]^L_{\mathrm{E}} \\
&+ 5\kappa\, \epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{nrM}\, A_t^K\, [A_s, A_l]^L_{\mathrm{E}} \\
&- 10\kappa\, \epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, \partial_R B_{nrM}\, A_s^K\, [A_t, A_l]^L_{\mathrm{E}} \\
&- 3\kappa\, \epsilon^{tnrsl}\, d_{MNP}\;\; A_t^N\;\; \partial_n A_r^M\;\; \partial_s A_l^P \\
&+ \frac{3\kappa}{4} \epsilon^{tnrsl}\, d_{MNP}\, A_t^N\;\; [A_n, A_r]^M_E\;\; \partial_s\, A_l^P \\
&- \frac{3\kappa}{2} \epsilon^{tnrsl}\, d_{MNP}\, A_n^N\;\; [A_t, A_r]^M_E\;\; \partial_s\, A_l^P \\
&+ \frac{3\kappa}{4} \epsilon^{tnrsl}\, d_{MNP}\, A_n^N\;\; [A_r, A_s]^M_E\;\; \partial_l\, A_t^P \\
&- \frac{3\kappa}{20} \epsilon^{tnrsl}\, d_{MNP}\, A_t^N\;\; [A_n, A_r]^M_E\;\; [A_s, A_l]^P_E \\
&+ \frac{3\kappa}{5} \epsilon^{tnrsl}\, d_{MNP}\, A_n^N\;\; [A_t, A_r]^M_E\;\; [A_s, A_l]^P_E
\end{aligned}
\tag{6.61}
$$

We then factor out the Lagrange multipliers in (6.61) for the non-topological terms and organise the terms according to the Lagrange multipliers as in (6.62).

$$
\begin{aligned}
&\dot{A}^N_n \cdot \Pi^n_N(A^N_n) - \mathcal{L}_{\mathrm{YM}} - E\,\mathcal{F}^M_{\alpha\beta} E^{\alpha\rho}\partial_M E^\beta_\rho - (\mathcal{L}_{\mathrm{top}} - \mathcal{L}_{\mathrm{BK}})\\
=&+\frac{e}{2\,N}\, g_{lm}\, M^{KL}\partial_L N^l\,\partial_K N^m \qquad (6.62)\\
&+N\cdot\Bigg(+\frac{N}{2\,e}\, g_{lm}\, M^{KL}\,\mathcal{P}^l_L(A)\,\mathcal{P}^m_K(A) + \frac{e}{4}\, M_{MN}\,\mathcal{F}^M_{rs}\,\mathcal{F}^N_{mn}\, g^{rm}\, g^{sn}\\
&\qquad - e\,\mathcal{F}^M_{mn}\, e^{ar}\,\partial_M e_r{}^b\, e_a{}^m\, e_b{}^n\Bigg)\\
&+N^n\cdot\Big(+\mathcal{F}^M_{nl}\,\mathcal{P}^l_M(A) + \partial_K\left(g_{mn}\, M^{KL}\mathcal{P}^m_L(A)\right)\Big)\\
&+A^M_t\cdot\Big(-\mathcal{D}_n\mathcal{P}^n_M(A) - 5\, d^{KLR}\, d_{MRS}\, A^S_n\,\partial_L\mathcal{P}^n_K(A)\Big)\\
&+15\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\,\partial_l\partial_R B_{nrM}\, A^K_s\, A^L_t\\
&-30\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\,\partial_R B_{nrM}\, A^K_t\,\partial_s A^L_l\\
&+5\kappa\,\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\,\partial_R B_{nrM}\, A^K_t\,[A_s, A_l]^L_{\mathrm{E}}\\
&-10\kappa\,\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\,\partial_R B_{nrM}\, A^K_s\,[A_t, A_l]^L_{\mathrm{E}}\\
&-3\kappa\,\epsilon^{tnrsl}\, d_{MNP}\;\, A^N_t\;\,\partial_n A^M_r\;\,\partial_s A^P_l\\
&+\frac{3\kappa}{4}\epsilon^{tnrsl}\, d_{MNP}\, A^N_t\;\,[A_n, A_r]^M_E\;\,\partial_s\, A^P_l\\
&-\frac{3\kappa}{2}\epsilon^{tnrsl}\, d_{MNP}\, A^N_n\;\,[A_t, A_r]^M_E\;\,\partial_s\, A^P_l\\
&+\frac{3\kappa}{4}\epsilon^{tnrsl}\, d_{MNP}\, A^N_n\;\,[A_r, A_s]^M_E\;\,\partial_l\, A^P_t\\
&-\frac{3\kappa}{20}\epsilon^{tnrsl}\, d_{MNP}\, A^N_t\;\,[A_n, A_r]^M_E\;\,[A_s, A_l]^P_E\\
&+\frac{3\kappa}{5}\epsilon^{tnrsl}\, d_{MNP}\, A^N_n\;\,[A_t, A_r]^M_E\;\,[A_s, A_l]^P_E\\
&+10\, d^{MKL}\, B_{tnL}\,\partial_K\mathcal{P}^n_M(A)\\
&+30\kappa\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, B_{tnM}\,\partial_R(A^K_r\,\partial_s A^L_l)\\
&-10\kappa\,\epsilon^{tnrsl}\, d^{MNR}\, d_{NKL}\, B_{tnM}\,\partial_R(A^K_r\,[A_s, A_l]^L_{\mathrm{E}})
\end{aligned}
$$

In the form of (6.62) we recognise the quadratic one-form field strength $\mathcal{F}^2$ and momenta terms $\mathcal{P}^2$, as well as the spatial Einstein-Hilbert improvement term, which will become part of the Hamilton constraint. Furthermore we find the expected diffeomorphism term $\mathcal{F}\mathcal{P}$ and the additional $+\partial_K\left(g_{mn}\, M^{KL}\mathcal{P}^m_L(A)\right)$ term which was already mentioned above.

The last three terms of (6.62) can be combined with the B_{tnM} term from section 6.5.3 into $\mathcal{H}_{\mathrm{S1}}$ as defined by (6.63).

$$
+B_{tlM}\cdot\left[10\, d^{MKL}\partial_K\left(\mathcal{P}^l_L - \kappa\,\epsilon^{lmnr}\,\mathcal{H}_{mnrL}\right)\right] =: +B_{tlM}\cdot(\mathcal{H}_{\mathrm{S1}})^{lM} \qquad (6.63)
$$

In analogy to the eponymous secondary constraints from the model of section 4.4, the $\mathcal{H}_{\mathrm{S1}}$ will be identified as secondary constraints that generate part of the tensor gauge transformations.

Finally we should also factor out all the remaining Lagrange multipliers of the topological terms. This expression is however not particularly interesting and makes the Hamiltonian much more complicated. What makes this expression complicated are in particular the terms with a time index in an E-bracket $[A_t, A_r]_E^M$, which generate many additional terms when factoring out the A_t. Hence we only state the topological terms in this form in the final form of the Hamiltonian topological term (6.68).

6.6 Canonical Hamiltonian of the $\mathrm{E}_{6(6)}$ ExFT

We can now combine the results of the partial computations from section 6.5 to piece together the Legendre transformations (6.42). We find that the full canonical Hamiltonian $\mathcal{H}_{\mathrm{ExFT}}$ of the (bosonic) $\mathrm{E}_{6(6)}$ exceptional field theory is give by (6.64).

$$
\begin{aligned}
\mathcal{H}_{\mathrm{ExFT}} = &+ N \cdot \Big[+ \frac{1}{4e}\Pi_{ab}(e)\,\Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\,\hat{R} + e\,V_{\mathrm{HP}} \\
&\qquad + \frac{3}{2e}\Pi^{MN}(M)\,\Pi^{RS}(M)\,M_{MR}\,M_{NS} - \frac{e}{24}g^{kl}\,\mathcal{D}_k M_{MN}\,\mathcal{D}_l M^{MN} \\
&\qquad + \frac{e}{4}M_{MN}\,g^{rm}\,g^{sn}\,\mathcal{F}_{rs}^M\,\mathcal{F}_{mn}^N + \frac{1}{2e}g_{lm}\,M^{KL}\,\mathcal{P}_L^l\,\mathcal{P}_K^m \Big] \\
&+ N^n \cdot \Big[+ 2\,\Pi^m{}_a(e)\,\mathcal{D}_{[n}e_{m]a} - e_{na}\,\mathcal{D}_m\Pi^m{}_a(e) \\
&\qquad + \frac{1}{2}\Pi^{MN}(M)\,\mathcal{D}_n M_{MN} \\
&\qquad + \mathcal{F}_{nl}^M\mathcal{P}_M^l + \partial_M\left(g_{mn}\,M^{MN}\,\mathcal{P}_N^m\right) \Big] \\
&+ A_t^M \cdot \Big[- \mathcal{D}_l\mathcal{P}_M^l - 5\,d^{NLS}\,d_{MNK}\,A_m^K\partial_S\mathcal{P}_L^m + (\mathcal{H}_{top})_M \\
&\qquad + \Pi^m{}_a(e)\,\partial_M e_{ma} - \frac{1}{3}\partial_M\Pi(e) \\
&\qquad + \frac{1}{2}\,\Pi^{KL}(M)\,\partial_M M_{KL} - 6\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left(\Pi^{KL}(M)\,M_{RL}\right) \Big] \\
&+ B_{tlM} \cdot \Big[+ 10\,d^{MKL}\partial_K\left(\mathcal{P}_L^l - \kappa\,\epsilon^{lmnr}\,\mathcal{H}_{mnrL}\right) \Big] \\
&+ \dot{N}\cdot\Pi(N) + \dot{N}_a\cdot\Pi^a(N_a) + \dot{A}_t^M\cdot\Pi_M(A_t) + \dot{B}_{tnN}\cdot\Pi^{nN}(B_{tnN})
\end{aligned}
\tag{6.64}
$$

In (6.64) the Hamiltonian is written in the form where the Lagrange multipliers N, N^n, A_t^M, B_{tlM} are factored out, which is useful because it makes the secondary constraints apparent.

The improved spatial Ricci scalar $\hat{R}$ in (6.64) is defined by (6.65), which we can compare to (3.99).

$$
-e\,\hat{R} := -e\,R_4 - e\,\mathcal{F}_{mn}^M\,e_a{}^m e_b{}^n\,(e^{ra}\partial_M e_r{}^b) \tag{6.65}
$$

The scalar potential of the Hamiltonian V_{HP} is defined by (6.66).

$$
N\,e\,V_{\mathrm{HP}} := N\,e\,V_{\mathrm{pot}} + \frac{e}{2N}g_{mn}M^{MN}\partial_M N^n\,\partial_N N^m \tag{6.66}
$$

As we can see in (6.66), V_{HP} is given by the Lagrangian scalar potential V_{pot} plus the contribution that originates from the interplay of the Einstein-Hilbert improvement term and the Yang-Mills term in section 6.5.4. This term cancels overall and the Hamiltonian scalar potential is explicitly given by (6.67).

$$\begin{aligned}+e\,V_{\mathrm{HP}} =& -\frac{e}{4}M^{MN}\partial_M g^{mn}\,\partial_N g_{mn} - \frac{1}{e}M^{MN}\partial_M e\,\partial_N e \\ & -\frac{e}{24}M^{MN}\partial_M M^{KL}\,\partial_N M_{KL} + \frac{e}{2}M^{MN}\partial_M M^{KL}\,\partial_L M_{NK} \\ & +\partial_M\partial_N M^{MN}\,e + 2\,M^{MN}\,\partial_M\partial_N e + 2\,\partial_M M^{MN}\,\partial_N e\end{aligned} \tag{6.67}$$

The Hamiltonian topological term $\mathcal{H}_{top}$ is defined by (6.68).

$$\begin{aligned}&(\mathcal{H}_{top})_M := \\ &+\frac{1}{2}\,\Pi^{lnN}(B)\,\partial_M B_{lnN} - 3\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left(\Pi^{lnK}(B)\,B_{lnR}\right) - \frac{1}{3}\,\partial_M\left(B_{mnN}\Pi^{mnN}(B)\right) \\ &-\frac{1}{2}\,(\mathcal{H}_{\mathrm{P2}})^{lnN}\,\partial_M B_{lnN} + 3\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left((\mathcal{H}_{\mathrm{P2}})^{lnK}\,B_{lnR}\right) + \frac{1}{3}\,\partial_M\left(B_{mnN}(\mathcal{H}_{\mathrm{P2}})^{mnN}\right) \\ &-3\kappa\,\epsilon^{tlmnr}\,d_{MNP}\,\partial_l A_m^N\,\partial_n A_r^P \\ &-15\kappa\,\epsilon^{tlmnr}\,d^{SRN}\,d_{MNK}\,\partial_l\partial_S B_{mnR}\,A_r^K - 30\kappa\,\epsilon^{tlmnr}\,d^{SRN}\,d_{MNK}\,\partial_S B_{mnR}\,\partial_l\,A_r^K \\ &+5\kappa\epsilon^{tlmnr}\,d^{SRN}\,d_{MNK}\,[A_l,A_m]_E^K\,\partial_S B_{nrR} - 20\kappa\epsilon^{tlmnr}d^{SRN}\,d_{QNK}\,A_l^K\,\partial_M A_m^Q\,\partial_S B_{nrR} \\ &+100\kappa\,\epsilon^{tlmnr}\,d^{NKT}\,d^{QRS}\,d_{MNL}\,d_{TPQ}\,A_l^P\,\partial_K A_m^L\,\partial_S B_{nrR} \\ &+\frac{3}{2}\kappa\,\epsilon^{tlmnr}\,d_{MNK}\,\partial_l A_m^N\,[A_n,A_r]_E^K - 3\kappa\,\epsilon^{tlmnr}\,d_{QNK}\,A_l^Q\,\partial_m A_n^N\,\partial_M A_r^K \\ &+15\kappa\,\epsilon^{tlmnr}\,d^{NRS}\,d_{MNK}\,d_{RLP}\,A_l^L\,\partial_S A_m^K\,\partial_n A_r^P \\ &-\frac{3}{2}\kappa\,\epsilon^{tlmnr}\,d_{MNP}\,A_l^N\,\partial_m A_n^S\,\partial_S A_r^P - \frac{3}{2}\kappa\,\epsilon^{tlmnr}\,d_{MNP}\,A_l^N\,A_n^S\,\partial_m\partial_S A_r^P \\ &+\frac{15}{2}\kappa\,\epsilon^{tlmnr}\,d^{PXS}\,d_{MNP}\,d_{XYZ}\,A_l^N\,\partial_m A_n^Y\,\partial_S A_r^Z \\ &+\frac{15}{2}\kappa\,\epsilon^{tlmnr}\,d^{PXS}\,d_{MNP}\,d_{XYZ}\,A_l^N\,A_n^Y\,\partial_m\partial_S A_r^Z \\ &-\frac{3}{20}\kappa\,\epsilon^{tlmnr}\,d_{MNK}\,[A_l,A_m]_E^N\,[A_n,A_r]_E^K + \frac{6}{5}\kappa\,\epsilon^{tlmnr}\,d_{QNK}\,A_l^Q\,\partial_M A_m^N\,[A_n,A_r]_E^K \\ &-6\kappa\,\epsilon^{tlmnr}\,d^{NRS}\,d_{MNK}\,d_{RLQ}\,A_l^L\,\partial_S A_m^K\,[A_n,A_r]_E^Q\end{aligned} \tag{6.68}$$

Every κ-dependent term that originates from the Lagrangian topological term $\mathcal{L}_{top}$ (3.135) goes either into the Hamiltonian topological term $\mathcal{H}_{top}$, into the modified momenta $\mathcal{P}_N^n$ or into the two-form field strength $\mathcal{H}_{klmN}$. The great complexity of the Hamiltonian topological term $\mathcal{H}_{top}$ is inherited from the complexity of the Lagrangian topological term $\mathcal{L}_{top}$ and exacerbated by the need to factor out the Lagrange multipliers A_t^M. By definition every term in $\mathcal{H}_{top}$ is linear in the coefficient of the Lagrangian topological term κ, although this coefficient is hidden in the terms coming from the two-form kinetic term due to the use of the primary constraints $\mathcal{H}_{\mathrm{P2}}$ (cf. section 6.5.3). $\mathcal{H}_{top}$ is analogous to the much simpler topological F^2 θ-term in the Gauß constraint (5.34) in the Hamiltonian formulation of the $\mathrm{E}_{6(6)}$ invariant five-dimensional ungauged maximal supergravity of chapter 5. In ExFT this "F^2" term can be found in the third line of (6.68), which is the only term that survives in the trivial solution of the section condition because it does not depend on any internal derivatives. We cannot express it in terms of the field strength in ExFT because the non-integral topological term of ExFT is not manifestly covariant.

It is instructive to further compare the $E_{6(6)}$ ExFT Hamiltonian (6.64) to the Hamiltonian $\mathcal{H}_{5D}$ (5.19) of five-dimensional $E_{6(6)}$ invariant supergravity. When the trivial solution of the section condition ($\partial_M = 0\ \forall M$) is applied to the ExFT Hamiltonian (6.64), we find that it reduces to the Hamiltonian $\mathcal{H}_{5D}$ (6.64) of the manifestly $E_{6(6)}$ invariant five-dimensional supergravity as expected. Every term of the five-dimensional Hamiltonian $\mathcal{H}_{5D}$ (5.19) can be found in the ExFT Hamiltonian (6.64), but the (external) derivatives, the one-form field strength and the Ricci scalar are replaced by the covariantised and improved expressions $\mathcal{D}_\mu$, $\mathcal{F}^M_{\mu\nu}$ and $\hat{R}$ (with the exception of the topological term). Additionally there are purely internal terms that completely vanish in the trivial solution of the section condition and that have no analogue in five dimensions, which includes the scalar potential terms V_{HP}, the term $+N^n\,\partial_M\left(g_{mn}\,M^{MN}\,\mathcal{P}^m_N\right)$, which we discuss in detail in section 6.9.3, as well as all of the B_{tlM} dependent terms and most of the A^M_t dependent terms. In the five-dimensional Hamiltonian $\mathcal{H}_{5D}$ (5.19) the A^M_t dependent terms are the Gauß constraints and generated the abelian $\text{U}(1)^{27}$ gauge transformations. In the ExFT Hamiltonian (6.64) there are far more A^M_t dependent terms and the analogue expression is significantly more complicated due to the covariant derivatives $\mathcal{D}_\mu$ in the Lagrangian and due to the complicated ExFT topological term. We will see in section 6.9.2 that the A^M_t dependent terms are the constraints which generate the generalised exceptional diffeomorphisms. There is no analogue for the B_{tlM} dependent terms in $\mathcal{H}_{5D}$ (5.19) because there is no two-form in the field content of the five-dimensional theory — we will find that these constraints generate a part of the tensor gauge transformations.

We can appreciate how much simpler the ExFT Hamiltonian is due to the use of the modified canonical momenta $\Pi^m_N(A)$ if we insert the definition (6.21) of the modified momenta back into the Hamiltonian (6.64) and see how many additional topological constraints are generated. Looking at (6.64) the $\mathcal{P}^m_N$ (6.21) hence seem to be the best and most natural variables to use.

6.7 Fundamental Poisson brackets

Next we need to define the fundamental Poisson brackets before we can begin to construct the complete set of canonical constraints. Here we take $X_1 = (x_1, Y_1)$ to mean the tuple of the four spatial external and the internal coordinates and define $X_1 - X_2 = (x_1 - x_2, Y_1 - Y_2)$ to be the difference between two such tuples. The non-vanishing equal-time fundamental Poisson brackets are then defined as listed below.

$$\{N(X_1), \Pi(N)(X_2)\} = \delta^{(4+27)}(X_1 - X_2) \tag{6.69}$$

$$\{N^n(X_1), \Pi_m(N^k)(X_2)\} = \delta^n_m \delta^{(4+27)}(X_1 - X_2) \tag{6.70}$$

$$\{e_n{}^a(X_1), \Pi^m{}_b(e)(X_2)\} = \delta^m_n \delta^a_b \delta^{(4+27)}(X_1 - X_2) \tag{6.71}$$

$$\{A^M_t(X_1), \Pi_N(A^K_t)(X_2)\} = \delta^M_N \delta^{(4+27)}(X_1 - X_2) \tag{6.72}$$

$$\{A^M_m(X_1), \Pi^n_N(A^K_k)(X_2)\} = \{A^M_m(X_1), \mathcal{P}^n_N(X_2)\} \tag{6.73}$$

$$= \delta^M_N \delta^n_m \delta^{(4+27)}(X_1 - X_2) \tag{6.74}$$

$$\{B_{tlR}(X_1), \Pi^{tnS}(B_{tqQ})(X_2)\} = \delta^n_l \delta^S_R\, \delta^{(4+27)}(X_1 - X_2) \tag{6.75}$$

$$\{B_{klR}(X_1), \Pi^{mnS}(B_{pqQ})(X_2)\} = (\delta^m_k \delta^n_l - \delta^m_l \delta^n_k)\, \delta^S_R\, \delta^{(4+27)}(X_1 - X_2) \tag{6.76}$$

$$\{M_{MN}(X_1), \Pi^{PQ}(M)(X_2)\} = \left(\delta^P_M\, \delta^Q_N + \delta^P_N\, \delta^Q_M\right) \delta^{(4+27)}(X_1 - X_2) \tag{6.77}$$

As we can see from the Poisson bracket (6.73) the modified momenta $\mathcal{P}^n_N(A)$ satisfy the same Poisson bracket relation with the one-forms as the original one-form momenta, because the topological terms that were subtracted in (6.21) only depend on the differential forms. Because (6.21) is not a canonical transformation the modified momenta do however not Poisson-commute among themselves $\{\mathcal{P}^l_L(A), \mathcal{P}^k_K(A)\} \neq 0$ or with the two-form momenta $\{\mathcal{P}^l_L(A), \Pi^{mnS}(B)\} \neq 0$.

Because we use the implicit formalism for the $E_{6(6)}/USp(8)$ coset constraints (cf. section 4.3) the relation (6.77) is simply the fundamental Poisson bracket of a generic scalar matrix and there is in particular no coset projector term.

6.8 Canonical constraints

We now construct the secondary constraints that arise as the consistency conditions of the primary constraints from section 6.4. Here we follow the procedure described in section 4.1.

6.8.1 Total Hamiltonian

The consistency of the primary constraints needs to be verified with respect to the total Hamiltonian (6.78), which consists of the canonical Hamiltonian plus a general phase space linear combination of the primary canonical constraints.

$$\begin{aligned}\mathcal{H}_{\text{ExFT-Total}} := \mathcal{H}_{\text{ExFT}} &+ u_0 \cdot \Pi(N) + (u_1)_a \cdot \Pi^a(N_a) + (u_2)^{ab} \cdot L_{ab} \\ &+ (u_3)^M \cdot \Pi^t_M(A) + (u_4)_{lN} \cdot (\mathcal{H}_{\text{P1}})^{lN} + (u_5)_{slN} \cdot (\mathcal{H}_{\text{P2}})^{slN}\end{aligned} \tag{6.78}$$

The consistency algorithm is simplified by first computing the algebra that the primary constraints form under the fundamental Poisson brackets from section 6.7. As expected the Lorentz constraints form the Lorentz subalgebra, which is again of the form (5.74). One can check that every other Poisson bracket among the primary constraints vanishes. The only non-trivial primary constraint algebra relations are those among the two-form primary constraints, which we have already seen in the model of section 4.4.2.

$$\{(\mathcal{H}_{\text{P1}})^{kK}, (\mathcal{H}_{\text{P1}})^{lL}\} = 0 \tag{6.79}$$

$$\{(\mathcal{H}_{\text{P1}})^{kK}, (\mathcal{H}_{\text{P2}})^{mnM}\} = 0 \tag{6.80}$$

$$\{(\mathcal{H}_{\text{P2}})^{klK}, (\mathcal{H}_{\text{P2}})^{mnM}\} = 0 \tag{6.81}$$

6.8.2 Secondary constraints

The condition for the consistency of the primary constraints (cf. equation (4.9) in section 4.1) tells us that the primary constraints have to be conserved in time, under the time evolution generated by the total Hamiltonian, for the formalism to be consistent. The consistency of each of the shift type primary constraints $\Pi(N)$, $\Pi^n(N_n)$, $\Pi_M(A^M_t)$ and $(\mathcal{H}_{P1})^{lM}(B) = \Pi^{tlM}(B)$ respectively requires the following expressions to be secondary canonical constraints: the Hamilton constraint $\mathcal{H}_{\text{Ham}}$ (6.82), the (external) diffeomorphism constraints $\mathcal{H}_{\text{Diff}}$ (6.83), the (internal) generalised exceptional diffeomorphism constraints $\mathcal{H}_{\text{GD}}$ (6.84) and the two-form tensor gauge constraints

$\mathcal{H}_{\mathrm{S1}}$ (6.85).

$$\begin{aligned}
\mathcal{H}_{\mathrm{Ham}} :=& + \frac{1}{4e}\Pi_{ab}(e)\ \Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\ \hat{R} + e\,V_{\mathrm{HP}} &(6.82)\\
&+ \frac{3}{2e}\Pi^{MN}(M)\ \Pi^{RS}(M)\ M_{MR}\ M_{NS} - \frac{e}{24}g^{kl}\ \mathcal{D}_k M_{MN}\ \mathcal{D}_l M^{MN}\\
&+ \frac{e}{4}M_{MN}\ g^{rm}\ g^{sn}\ \mathcal{F}^M_{rs}\ \mathcal{F}^N_{mn} + \frac{1}{2e}g_{lm}\ M^{KL}\ \mathcal{P}^l_L\ \mathcal{P}^m_K\\
(\mathcal{H}_{\mathrm{Diff}})_n :=& + 2\ \Pi^m{}_a(e)\ \mathcal{D}_{[n}e_{m]a} - e_{na}\ \mathcal{D}_m\Pi^m{}_a(e) &(6.83)\\
&+ \frac{1}{2}\Pi^{MN}(M)\ \mathcal{D}_n M_{MN}\\
&+ \mathcal{F}^M_{nl}\mathcal{P}^l_M + \partial_M\left(g_{mn}\,M^{MN}\,\mathcal{P}^m_N\right)\\
(\mathcal{H}_{\mathrm{GD}})^M :=& - \mathcal{D}_l\mathcal{P}^l_M - 5\,d^{NLS}\,d_{MNK}\,A^K_m\partial_S\mathcal{P}^m_L + (\mathcal{H}_{top})_M &(6.84)\\
&+ \Pi^m{}_a(e)\,\partial_M e_{ma} - \frac{1}{3}\partial_M\Pi(e)\\
&+ \frac{1}{2}\,\Pi^{KL}(M)\,\partial_M M_{KL} - 6\,\mathbb{P}^R{}_K{}^S{}_M\,\partial_S\left(\Pi^{KL}(M)\,M_{RL}\right)\\
(\mathcal{H}_{\mathrm{S1}})^{lM} :=& + 10\,d^{MKL}\partial_K\left(\mathcal{P}^l_L - \kappa\,\epsilon^{lmnr}\,\mathcal{H}_{mnrL}\right) &(6.85)
\end{aligned}$$

The above secondary constraints make up the main part of the canonical Hamiltonian (6.64) and on the primary constraint surface the Hamiltonian only consists of these secondary constraints. Because ExFT is a generally covariant the Hamiltonian is weakly vanishing.

As was already discussed for the five-dimensional supergravity in section 5.2.5, the Lorentz constraints do not require any secondary constraints.

We can see that the secondary constraints $\mathcal{H}_{\mathrm{S1}}$ (6.85) are the covariantised ExFT version of the eponymous constraints (4.61) of the two-form model theory from section 4.4, which originated from the consistency condition (4.68). Moreover we should note the similarity of the covariantised constraints (6.85) to the Lagrangian equations of motion of the two-forms (3.113), which act as the duality relation between the one- and two-forms.

Lastly we have to examine the consistency of the primary two-form constraints $\mathcal{H}_{\mathrm{P2}}$ (6.38), which are not of the simple shift type form. In section 4.4 we found that the consistency of the analogous constraints $\mathcal{H}_{\mathrm{P2}}$ of the model theory requires the secondary constraints $\mathcal{H}_{\mathrm{S2}}$ (4.72) because of the non-vanishing Poisson bracket relation $\{\mathcal{H}_{\mathrm{P2}}, \mathcal{H}_{\mathrm{S1}}\} \neq 0$. The primary constraints $\mathcal{H}_{\mathrm{P2}}$ Poisson-commute with every other primary constraint and hence their time evolution as generated by the total Hamiltonian is identical to the time evolution generated by the canonical Hamiltonian. The consistency conditions for $\mathcal{H}_{\mathrm{P2}}$ can thus be written as (6.86).

$$0 \overset{!}{=} \{(\mathcal{H}_{\mathrm{P2}})^{mnM}, \mathcal{H}_{\text{ExFT-Total}}\} = \{(\mathcal{H}_{\mathrm{P2}})^{mnM}, \mathcal{H}_{\mathrm{ExFT}}\} \tag{6.86}$$

What makes the consistency condition (6.86) of ExFT more complicated than the analogous condition in the model theory from section 4.4 is the fact that there are other (secondary) constraints in the ExFT Hamiltonian $\mathcal{H}_{\mathrm{ExFT}}$, all of which depend on the two-forms. This two form dependence is hidden inside the topological term, the modified one-form momenta or the Stückelberg coupling in the one-form field strength. Furthermore the secondary constraints do not Poisson-commute with $\mathcal{H}_{\mathrm{P2}}$. Because the Lagrangian kinetic term of the two-forms is linear in the time derivative the Hamiltonian does not depend on the two-form momenta $\Pi^{klL}(B)$. Hence only the $\Pi^{mnM}(B)$ part of the constraints $\mathcal{H}_{\mathrm{P2}}$ is relevant to the calculation of (6.86). This allows us to express (6.86) in terms of the transformation (6.87) of the momenta $\Pi^{mnM}(B)$ generated by the canonical Hamiltonian.

$$\{(\mathcal{H}_{\mathrm{P2}})^{mnM}, \mathcal{H}_{\mathrm{ExFT}}\} = \{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{ExFT}}\} \stackrel{!}{=} 0 \tag{6.87}$$

In the model theory of section 4.4 the consistency condition which is directly analogous to (6.87) implies the secondary constraints (4.72). However in ExFT there are contributions to (6.87) from every secondary constraint. The Lagrange multipliers of the secondary constraints in the Hamiltonian are independent and hence we should be able to split up (6.87) into the independent consistency conditions (6.88), (6.89), (6.90) and (6.91), where we use the smeared version of the secondary constraints.

$$\{\mathcal{H}_{\mathrm{P2}}[\lambda], \mathcal{H}_{\mathrm{Ham}}[N]\} = \{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{Ham}}[N]\} \neq 0 \tag{6.88}$$
$$\{\mathcal{H}_{\mathrm{P2}}[\lambda], \mathcal{H}_{\mathrm{Diff}}[N^l]\} = \{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{Diff}}[N^l]\} \neq 0 \tag{6.89}$$
$$\{\mathcal{H}_{\mathrm{P2}}[\lambda], \mathcal{H}_{\mathrm{GD}}[A_t^L]\} = \{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{GD}}[A_t^L]\} \neq 0 \tag{6.90}$$
$$\{\mathcal{H}_{\mathrm{P2}}[\lambda], \mathcal{H}_{\mathrm{S1}}[B_{tlL}]\} = \{\Pi^{mnM}(B), \mathcal{H}_{\mathrm{S1}}[B_{tlL}]\} \neq 0 \tag{6.91}$$

The above consistency conditions are simply the statement that the transformations of $\Pi^{mnM}(B)$ generated by these secondary constraints, with the transformation parameters given by the relevant Lagrange multipliers, should vanish. These conditions are in direct analogy to the second-class secondary $\mathcal{H}_{\mathrm{S2}}$ constraints of the model theory from section 4.4 and we should expect that they behave in way that is similar to the results of section 4.4.3. These second class consistency conditions should then in particular require the introduction of a Dirac bracket, which involves the same issues that were discussed in section 4.4.4. In section 6.9.5 we calculate the explicit transformations that are equivalent to (6.88), (6.89), (6.90) and (6.91).

The consistency of the secondary constraints that we have found does not require any further (tertiary) canonical constraints.

6.9 Canonical (gauge) transformations of the $\mathrm{E}_{6(6)}$ ExFT

In this section we calculate the transformations generated by the canonical constraints via the Poisson brackets on the canonical coordinates. As in the previous chapters we may think of the canonical (gauge) transformations generated by an integrated (first class) constraint $\mathcal{H}[\lambda]$ on any canonical coordinate X as the infinitesimal transformation $\delta_{\mathcal{H}[\lambda]}X = \{X, \mathcal{H}[\lambda]\}$. Without the full constraint algebra we do not know at this point which constraints are first class and which are second class. Hence we will intuitively use the term "gauge transformation" in this chapter for those canonical transformations which we can identify with the gauge transformations of the Lagrangian formulation of ExFT.

In the section 6.9.1 we briefly explain why it may be computationally advantageous to examine the (gauge) transformations in the "non-topological" limit of $\kappa = 0$.
Thereafter the canonical (gauge) transformations are computed and considered on a

constraint by constraint basis.

In the remainder of this chapter we generally use the smeared (or integrated) versions of the canonical constraints, which have been described in section 4.1, in order to avoid having to write derivatives of the Dirac delta distribution.

6.9.1 Gauge transformations at $\kappa = 0$

The complexity of the Hamiltonian topological term $\mathcal{H}_{top}$, defined by (6.68), is inherited from the Lagrangian topological term and poses a major computational challenge to the canonical analysis of the $E_{6(6)}$ ExFT. To simplify the canonical Hamiltonian we modified the canonical momenta of the one-forms in section 6.3 by subtracting all topological contributions. Because this modification is not a canonical transformation the modified momenta do not Poisson-commute $\{\mathcal{P}^n_N(A), \mathcal{P}^m_M(A)\} \neq 0$. This procedure is identical to the treatment of the topological term of the $E_{6(6)}$ invariant five-dimensional supergravity in chapter 5, but the topological term of the $E_{6(6)}$ ExFT is unfortunately far more complicated. Nonetheless we can proceed with the computations in the same manner as in chapter 5.
We first carry out the computation for the case $\kappa = 0$ and often we find that this expression already contains much of the relevant information about the overall result because only the topological contribution is missing. Next we proceed to compute the topological contributions which are linear in the coefficient κ. This is the full result, because the terms that are quadratic in κ can only consist of Poisson brackets that do not involve any canonical momenta and hence have to vanish.

For many computations the only difference between $\kappa = 0$ and $\kappa \neq 0$ is that the one-form momenta $\Pi^n_N(A)$ are replaced by the modified momenta $\mathcal{P}^n_N$ and in some cases the full result does not depend on κ at all. The transformations of the modified momenta $\mathcal{P}^n_N$ are however particularly difficult to compute for the case of $\kappa \neq 0$ and for some of these transformations we only give the result for the $\kappa = 0$ computation because the computation of the remaining terms becomes too difficult. What we are missing completely in the case of $\kappa = 0$ is the dynamics of the two-forms, which is purely topological.

We only consider the case $\kappa = 0$ as a *computational tool* that allows us to manage one of the main difficulties of this canonical analysis. Moreover this method allows us to formulate partial results for some of the most difficult calculations. It is very likely that the case $\kappa = 0$ has no meaningful physical interpretation and upon solution of the section condition it probably does not correspond to any physically meaningful theory.

It is useful to briefly examine the consequences of setting $\kappa = 0$. By definition the modified one-form momenta reduce to the canonical momenta $\mathcal{P}^n_N = \Pi^n_N(A)$ and the topological term in the Hamiltonian vanishes $\mathcal{H}_{top} = 0$. The two-form field strength $\partial_K \mathcal{H}_{klmN}$ term in the tensor gauge constraints $\mathcal{H}_{\mathrm{S}1}$ vanishes too. Although the dynamics of the two-forms is governed by a topological kinetic term, the two-form constraints $\mathcal{H}_{\mathrm{S}1}$ do not vanish completely due to the remaining B_{tlM} term that originates from the Stückelberg coupling term in the field strength $\mathcal{F}^M_{\mu\nu}$ of the generalised Yang-Mills term. The canonical Hamiltonian in the case $\kappa = 0$, denoted by $\mathcal{H}^{\kappa=0}_{\mathrm{ExFT}}$, is hence vastly simplified and explicitly given by (6.92).

$$
\begin{aligned}
\mathcal{H}^{\kappa=0}_{\text{ExFT}} = &+ N \cdot \Big[+ \frac{1}{4e}\Pi_{ab}(e)\, \Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\, \hat{R} + e\, V_{\text{HP}} \\
&\qquad + \frac{3}{2e}\Pi^{MN}(M)\, \Pi^{RS}(M)\, M_{MR}\, M_{NS} - \frac{e}{24}g^{kl}\, \mathcal{D}_k M_{MN}\, \mathcal{D}_l M^{MN} \\
&\qquad + \frac{e}{4}M_{MN}\, g^{rm}\, g^{sn}\, \mathcal{F}^M_{rs}\, \mathcal{F}^N_{mn} + \frac{1}{2e}g_{lm}\, M^{KL}\, \Pi^l_L(A)\, \Pi^m_K(A) \Big] \\
&+ N^n \cdot \Big[+ 2\, \Pi^m{}_a(e)\, \mathcal{D}_{[n}e_{m]a} - e_{na}\, \mathcal{D}_m \Pi^m{}_a(e) \\
&\qquad + \frac{1}{2}\Pi^{MN}(M)\, \mathcal{D}_n M_{MN} \\
&\qquad + \mathcal{F}^M_{nl}\, \Pi^l_M(A) + \partial_M \left(g_{mn}\, M^{MN}\, \Pi^m_N(A) \right) \Big] \\
&+ A^M_t \cdot \Big[- \mathcal{D}_l \Pi^l_M(A) - 5\, d^{NLS}\, d_{MNK}\, A^K_m \partial_S \Pi^m_L(A) \\
&\qquad + \Pi^m{}_a(e)\, \partial_M e_{ma} - \frac{1}{3}\partial_M \Pi(e) \\
&\qquad + \frac{1}{2}\, \Pi^{KL}(M)\, \partial_M M_{KL} - 6\, \mathbb{P}^R{}_K{}^S{}_M\, \partial_S \left(\Pi^{KL}(M)\, M_{RL} \right) \Big] \\
&+ B_{tlM} \cdot \Big[+ 10\, d^{MKL}\, \partial_K \Pi^l_L(A) \Big] \\
&+ \dot{N} \cdot \Pi(N) + \dot{N}_a \cdot \Pi^a(N_a) + \dot{A}^M_t \cdot \Pi_M(A_t)
\end{aligned}
\tag{6.92}
$$

In the following we will add a $\kappa = 0$ to the name of the (secondary) constraints, e.g. $\mathcal{H}^{\kappa=0}_{\text{S1}}$, to indicate that the above version of the constraints is meant.

6.9.2 Generalised exceptional diffeomorphisms

We find that the generalised diffeomorphism constraints $\mathcal{H}_{\text{GD}}$ (6.84) (mainly) generate the generalised exceptional diffeomorphisms in the form of the generalised Lie derivative, which includes the correct generalised weight terms. The generalised Lie derivative $\delta_\Lambda = \mathbb{L}_\Lambda$ is the full result for the transformation of the spatial vielbein, the scalar fields and their conjugate canonical momenta, which agrees with the transformations described in the Lagrangian formulation. The structure of the generalised Lie derivative is already apparent in the form of the constraints (6.84).

$$\{e_{na}, \mathcal{H}_{\text{GD}}[\zeta]\} = \mathbb{L}_\zeta e_{na} \tag{6.93}$$
$$\{\Pi^n{}_a(e), \mathcal{H}_{\text{GD}}[\zeta]\} = \mathbb{L}_\zeta \Pi^n{}_a(e) \tag{6.94}$$
$$\{M_{MN}, \mathcal{H}_{\text{GD}}[\zeta]\} = \mathbb{L}_\zeta M_{MN} \tag{6.95}$$
$$\{\Pi^{MN}(M), \mathcal{H}_{\text{GD}}[\zeta]\} = \mathbb{L}_\zeta \Pi^{MN}(M) \tag{6.96}$$

As in the Lagrangian formalism the situation is more complicated for the transformations of the generalised differential forms.

The relevant part of the constraints for the transformation of the one-forms are the momentum terms from the expression (6.60). We can see from the transformation (6.97) that the $\mathcal{H}_{\text{GD}}$ constraints generate a generalised diffeomorphism transformation in the one-forms, as represented by the generalised Lie derivative acting on the one-forms, but we also find that they generate further terms.

$$
\begin{aligned}
\{A_n^N, \mathcal{H}_{\rm GD}[\zeta]\} &= \mathbb{L}_\zeta A_n^N + \partial_n \zeta^N - 5\, d^{NLR}\, \partial_L (d_{RMK}\, \zeta^M A_n^K) && (6.97)\\
&= \mathcal{D}_n \zeta^N + 5\, d^{NLR}\, \partial_L (d_{RMK}\, \zeta^M A_n^K) && (6.98)\\
&= \mathcal{D}_n \zeta^N + \delta_{\mathcal{H}_{S1}[-\frac{1}{2}\, d_{\cdot MK}\, \zeta^M A_\cdot^K]} (A_n^N) && (6.99)
\end{aligned}
$$

We can recognise the second term in (6.97) as the abelian $\mathrm{U}^{27}(1)$ transformation (5.53), which we know from the five-dimensional supergravity from chapter 5 and this is the only part of this transformations that exists in the trivial solution of the section condition. The additional momentum term in (6.60), which was found during the Legendre transformation, is responsible for generating the last term in the transformation (6.97). To rewrite the transformation (6.97) we can use the symmetrisation of the ExFT Dorfman bracket relation (3.82) to find the useful identity (6.100).

$$
\mathbb{L}_\zeta A_n^N = -\mathbb{L}_{A_n} \zeta^N + 10\, d^{NLR}\, d_{RMK}\, \partial_L (\zeta^M A_n^K) \qquad (6.100)
$$

By combining the definition of the covariant derivative with the identity (6.100) we are able to rewrite the transformation (6.97) in the form of (6.98). Instead of cancelling the trivial term coming from the identity (6.100) the extra term in the transformation (6.97) simply switches sign. As we will see later, the trivial extra term in (6.98) should be thought of as a tensor gauge transformation, as generated by the $\mathcal{H}_{S1}$ constraints on the one-forms. We can thus write the transformation as (6.99) and we find that this expression agrees with the Lagrangian gauge transformation (3.86), up to the tensor gauge transformation term.

Equation (6.101) gives the transformation of the (original) canonical one-form momenta $\Pi_N^n(A)$ at $\kappa = 0$.

$$
\begin{aligned}
\{\Pi_N^n(A), \mathcal{H}_{\rm GD}[\zeta]\} &= \mathbb{L}_\zeta \Pi_N^n(A) - 5\, d^{PKL} d_{NMP}\, \zeta^M\, \partial_K \Pi_L^n(A) && (6.101)\\
&= \mathbb{L}_\zeta \Pi_N^n(A) - \frac{1}{2}\, d_{NMP}\, \zeta^M\, (\mathcal{H}_{\rm S1}^{\kappa=0})^{nP} && (6.102)
\end{aligned}
$$

In the transformation (6.101) we can find the generalised Lie derivative term, but there is again an additional term, which also originates from the second momentum term in (6.60). We can identify the additional term in (6.101) as the $\kappa = 0$ version of the $\mathcal{H}_{\rm S1}$ constraints (cf. equation (6.92) and we can rewrite the transformation as (6.102).

Due to the $\mathcal{P}_N^n$ terms in the constraints (6.84) and due to the complexity of the topological term the calculation of the full transformation of $\mathcal{P}_N^n$ is computationally exceedingly complicated. What we can say about the full transformation of $\mathcal{P}_N^n$ is that it should certainly reduce to the expression (6.102) for the case $\kappa = 0$. It should therefore be expected that some of the topological contributions arrange into the modified momenta $\mathcal{P}_N^n(A)$ and some into the full $\mathcal{H}_{\rm S1}$ constraint, which in particular includes the covariantisation terms in the two-form field strength. One might hope that the transformation of the form (6.103) is the full transformation and that all other contributions cancel, but this may be too optimistic and there may indeed be further transformations that appear in (6.103) which remain to be determined.

$$\{\mathcal{P}^n_N(A), \mathcal{H}_{\text{GD}}[\zeta]\} \stackrel{?}{=} \mathbb{L}_\zeta \mathcal{P}^n_N(A) - \frac{1}{2}\, d_{NMP}\, \zeta^M\, (\mathcal{H}_{\text{S1}})^{nP} \tag{6.103}$$

Moreover it may be interesting to notice the analogy between the appearance of the $\mathrm{U}(1)^{27}$ one-form gauge transformations in the canonical (external) diffeomorphism transformations of the ungauged maximal five-dimensional $\mathrm{E}_{6(6)}$ invariant supergravity and the appearance of the two-form gauge transformations in the canonical (internal) generalised exceptional diffeomorphisms of the ExFT, which we have seen above. This analogy can be seen more readily when comparing the explicit canonical transformations generated by the diffeomorphism constraints of five-dimensional supergravity on the one-forms (5.60) and their momenta (5.61) to the transformations generated by the generalised diffeomorphism constraints on the one-forms (6.99) and their momenta (6.102). What we can see in this comparison is that where a $\mathrm{U}(1)^{27}$ gauge transformation and the Gauß constraints (i.e. the constraints which generate the $\mathrm{U}(1)^{27}$ transformations) appear in the standard diffeomorphisms of the five-dimensional supergravity, respectively a tensor gauge transformation and the $\mathcal{H}_{\text{S1}}$ constraints appear in the generalised diffeomorphism transformations in the ExFT.

Because the Lagrangian kinetic term of the two-forms (6.45) is linear in the time derivative this term cancels in the Legendre transformation described in section 6.5.3. The canonical Hamiltonian hence cannot depend on the two-form canonical momenta $\Pi^{klM}(B)$, which is also illustrated by the Hamiltonian of the two-form model theory in section 4.4. The absence of the canonical momenta from the canonical Hamiltonian is a general feature of theories where the kinetic term only depends on a single time derivative. Moreover this implies that the two-forms cannot transform under any of the secondary constraints that make up the canonical Hamiltonian. The primary constraints $\mathcal{H}_{\text{P2}}$ (6.38), which are not part of the Hamiltonian, do however lead to non-vanishing transformations of the two-forms, as we have seen in section 4.4. By definition the two-form canonical momenta are contained in the constraints $\mathcal{H}_{\text{P2}}$, which directly relate the momenta to (the internal derivative of) the two-form components. We can make use of the primary constraints $\mathcal{H}_{\text{P2}}$ to insert the canonical momenta $\Pi^{klM}(B)$ into the two-form kinetic term, which is part of $\mathcal{H}_{GD}$, to rewrite it as in (6.47). If we then apply the primary constraints $\mathcal{H}_{\text{P2}} = 0$ we do indeed find that the (6.47) terms in $\mathcal{H}_{GD}$ generate the generalised Lie derivative acting on the two-forms as (6.104).

$$\{B_{mkZ}, \mathcal{H}_{\text{GD}}[\zeta]\} = \mathbb{L}_\zeta B_{mkZ} \quad \text{if } (\mathcal{H}_{\text{P2}}) = 0 \text{ is used, else } 0 \tag{6.104}$$

What is problematic about the transformation (6.104) is that we are not allowed to apply the canonical constraints inside the Poisson bracket, but this would be allowed inside the Dirac bracket [28]. From the model of section 4.4 we know that the two-form constraints are second class constraints and hence we should ultimately construct the Dirac bracket. It does however seem rather likely that the transformation (6.104) would take the same form in the Dirac bracket, perhaps with some additional gauge transformation terms being generated.

The transformation of the two-form momenta $\Pi^{mkZ}(B)$ that is generated by $\mathcal{H}_{GD}$ is given by (6.105).

$$
\begin{aligned}
\{\Pi^{mkZ}(B), \mathcal{H}_{\mathrm{GD}}[\zeta]\} = &-300\,\kappa\,\epsilon^{tmknr}\,d^{RNQ}\,d^{ZST}\,d_{MNT}\,\partial_S\left(\zeta^M\,\partial_R B_{nrQ}\right) \\
&- 15\kappa\,\epsilon^{tmkls}\,d^{ZNR}\,d_{NKM}\,\partial_l\partial_R\left(\zeta^M\,A_s^K\right) \\
&+ 30\kappa\,\epsilon^{tmklr}\,d^{ZNR}\,d_{NKT}\,\partial_R\left(\partial_L\zeta^T\,A_r^K\,A_l^L\right) \\
&- 10\kappa\,\epsilon^{tmklr}\,d^{ZNR}\,d_{NKT}\,\partial_R\left(\zeta^L\,A_r^K\,\partial_L A_l^T\right) \\
&- 150\kappa\,\epsilon^{tmklr}\,d^{ZNR}\,d_{NKT}\,d^{LTS}\,d_{PLQ}\,\partial_R\left(\partial_S\zeta^P\,A_r^K\,A_l^Q\right) \\
&+ 50\kappa\,\epsilon^{tmklr}\,d^{ZNR}\,d_{NQL}\,d^{LKX}\,d_{MYX}\,\partial_R\left(\zeta^M\,A_r^Q\,\partial_K A_l^Y\right) \\
&+ 5\kappa\,\epsilon^{tmklr}\,d^{ZNR}\,d_{NML}\,\partial_R\left(\zeta^M\,[A_l, A_r]_{\mathrm{E}}^L\right)
\end{aligned}
\tag{6.105}
$$

The transformation (6.105) has many topological contributions that originate from the two-form terms inside the topological term that is contained in the $\mathcal{H}_{GD}$ constraints. There is no obvious simplification for these terms, but we know from the consistency condition (6.90) that the expression (6.105) should itself be a canonical constraint if we replace the parameter by $\zeta^M = A_t^M$. This constraint is then the analogue of the $\mathcal{H}_{\mathrm{S2}}$ constraints that we found in the model theory in section 4.4. Without the explicit form of the Dirac bracket these constraints and the transformation (6.105) are very hard to interpret.
Nonetheless we find that if we again make use of the primary constraints $\mathcal{H}_{\mathrm{P2}}$ to insert the canonical momenta into $\mathcal{H}_{GD}$, then the term (6.47) will also generate the generalised Lie derivative of the two-form momenta $\mathbb{L}_\zeta \Pi^{mkZ}(B)$. For the momenta there will however be additional topological contributions in this transformation due to the topological (non-kinetic) two-form terms in the modified one-form momenta and inside the topological term $\mathcal{H}_{\mathrm{top}}$.

6.9.3 External diffeomorphisms

We find that the canonical external diffeomorphism constraints $\mathcal{H}_{\mathrm{Diff}}$ (6.83) (mainly) generate the covariantised external diffeomorphism transformations in the canonical variables.

When the constraints $\mathcal{H}_{\mathrm{Diff}}$ act on the spatial vielbein (6.106) and the scalar fields (6.108) we find that the resulting gauge transformations are identical to the covariantised versions of the standard diffeomorphism transformations, where all derivatives have been replaced by the (external) covariant derivatives. These transformations agree with the spatial parts of the Lagrangian gauge transformations (3.109) and (3.110) respectively.

$$
\{e_n{}^a, \mathcal{H}_{\mathrm{Diff}}[\xi]\} = +\,\xi^l\,\mathcal{D}_l e_n{}^a + \mathcal{D}_n\xi^l\,e_l{}^a \tag{6.106}
$$

$$
\begin{aligned}
\{\Pi^n{}_a(e), \mathcal{H}_{\mathrm{Diff}}[\xi]\} = &+\,\xi^l\,\mathcal{D}_l\Pi^n{}_a(e) - \mathcal{D}_l\xi^n\,\Pi^l{}_a(e) + \mathcal{D}_l\xi^l\,\Pi^n{}_a(e) \\
&+ 2\,e_{la}\,M^{MN}\,\partial_M\xi^{(l}\,\mathcal{P}_N^{n)}(A)
\end{aligned}
\tag{6.107}
$$

$$
\{M_{MN}, \mathcal{H}_{\mathrm{Diff}}[\xi]\} = +\,\xi^n\,\mathcal{D}_n M_{MN} \tag{6.108}
$$

$$
\begin{aligned}
\{\Pi^{MN}(M), \mathcal{H}_{\mathrm{Diff}}[\xi]\} = &+\,\xi^n\mathcal{D}_n\Pi^{MN}(M) + \mathcal{D}_n\xi^n\Pi^{MN}(M) \\
&- 2\,\partial_K\xi^m\,g_{mn}\,\mathcal{P}_L^n(A)\,M^{K(M}M^{N)L}
\end{aligned}
\tag{6.109}
$$

Similarly we find that the transformations of the canonical momenta of the vielbein (6.107) and of the scalar fields (6.109) are also given by the covariantised standard diffeomorphism transformations (with the appropriate external standard diffeomorphism weights, as listed in table 5.1), but there are additional $\partial_N \xi^n$ terms in these transformations. These $\partial_N \xi^n$ terms originate from the $-g_{lm}\, M^{KL} \mathcal{P}^l_L(A)\, \partial_K N^m$ term in the $\mathcal{H}_{\text{Diff}}$ constraints, which we identified in the Legendre transformation in section 6.5.4 as coming from the interplay of the Einstein-Hilbert improvement and the Yang-Mills terms. Because this term depends on the spatial components of all of the fields it leads to similar contributions in the transformations of all canonical momenta. And because this term in particular also depends on the modified one-form momenta it also contributes to the transformation of the one-forms, which is given by (6.110).

$$\{A^N_n, \mathcal{H}_{\text{Diff}}[\xi]\} = +\xi^m\, \mathcal{F}^N_{mn} - g_{mn}\, M^{MN}\, \partial_M \xi^m \tag{6.110}$$

When compared to the analogous transformation (5.59) in the five-dimensional supergravity from chapter 5 we can recognise the first term of (6.110) as the covariantised version of the standard diffeomorphism. Furthermore this term agrees with the spatial part of the first term in the Lagrangian gauge transformation (3.112). The second term in (6.110) also agrees with the spatial components of the $\partial_M \xi^m$ term of the Lagrangian transformation (3.112) in form, but surprisingly the signs of the $\partial_M \xi^m$ term do not agree. This point will be discussed further at the end of this section when we examine the origin of the $-g_{lm}\, M^{KL} \mathcal{P}^l_L(A)\, \partial_K N^m$ term in detail.

When the coordinate dependence of the gauge parameters of the external diffeomorphisms is restricted to depend only on the external coordinates, i.e. $\xi^n\left(x^\mu, Y^M\right) = \xi^n(x^\mu)$ or equivalently $\partial_N \xi^n = 0$, then the above transformations that are generated by the $\mathcal{H}_{\text{Diff}}[\xi]$ constraints are precisely the $\mathcal{D}$-covariantised version of the standard diffeomorphism transformations. External diffeomorphism transformations with the parameters $\xi^n(x^\mu)$ are a manifest symmetry of each of the terms in the $E_{6(6)}$ ExFT Lagrangian (3.98) [3, 25]. By contrast the external diffeomorphism transformations with parameters $\xi^n\left(x^\mu, Y^M\right)$ are not a manifest symmetry of (3.98), instead they connect the different terms of the Lagrangian and thereby the invariance of the action requires very specific coefficients in the Lagrangian which makes the $E_{6(6)}$ ExFT action unique (up to the overall scaling and the section condition) [3, 25]. Moreover it seems to be the case that the $\partial_M \xi^m$ terms in the transformations (6.107), (6.109) and (6.110) have a similar connecting function in the canonical formalism because these terms depend on different canonical variables and mix the coordinate dependence. We can therefore speculate that the closure of the (bosonic) canonical constraint algebra may require the cancellation of such mixing terms in the relations that concern the $\mathcal{H}_{\text{Diff}}[\xi]$ constraints which would in turn fix all coefficients in the canonical Hamiltonian.

The transformation of the (original) canonical one-form momenta $\Pi^n_N(A)$ is more complicated than the above transformations, because of the one-form dependent covariant derivative terms and the covariant one-form field strength in the diffeomorphism constraints (6.83). Under the $\kappa = 0$ version of the diffeomorphism constraints the momenta $\Pi^n_N(A)$ transform as (6.111).

$$
\begin{aligned}
\{\Pi^n_N(A), \mathcal{H}^{\kappa=0}_{\text{Diff}}[\xi]\} = &+ \xi^k\, \mathcal{D}_k \Pi^n_N(A) - \mathcal{D}_k \xi^n\, \Pi^k_N(A) + \mathcal{D}_k \xi^k\, \Pi^n_N(A) \\
&+ \xi^n \left(\mathcal{H}^{\kappa=0}_{\text{GD}}\right)_N + \frac{1}{2}\, d_{NKL}\, \xi^k\, A^K_k (\mathcal{H}^{\kappa=0}_{S1})^{nL} \\
&+ \partial_N \xi^k\, \Pi^n{}_a(e)\, e_k{}^a - \frac{1}{3}\, \partial_N \xi^n\, \Pi^k{}_a(e)\, e_k{}^a \\
&- \frac{1}{3}\, \partial_N \xi^n\, \Pi^{KL}(M)\, M_{KL} - \partial_K \xi^n\, \Pi^{KL}(M)\, M_{LN} \\
&+ 10\, d^{RSM}\, d_{KNR}\, \partial_S \xi^n\, \Pi^{KL}(M)\, M_{LM} \\
&+ 10\, d^{RSM}\, d_{KNR}\, \partial_S \xi^{[m}\, \Pi^{n]}_M(A)\, A^K_m
\end{aligned}
\tag{6.111}
$$

The covariantised standard Lie derivative of $\Pi^n_N(A)$ can be found in the first line of the transformation (6.111). The transformation (6.111) can be compared to the analogous transformation (5.61) of the canonical analysis of the five-dimensional supergravity from chapter 5 and we can see that the $\mathcal{H}^{\kappa=0}_{\text{GD}}$ term that appears in the transformation (6.111) is an extension of the $\text{U}(1)^{27}$ Gauß constraint term in the transformation (5.61). As we can see from (6.92) the $\mathcal{H}^{\kappa=0}_{S1}$ constraints consist only of a $\Pi^n_N(A)$ term. The $\Pi^n_N(A)$ terms in (6.111), some of which are hidden inside the $\mathcal{H}^{\kappa=0}_{\text{GD}}$ and $\mathcal{H}^{\kappa=0}_{S1}$ constraints originate from the transformation of the covariantised one-form field strength $\mathcal{F}^M_{mn}$. All of the remaining terms in (6.111) are of the $\partial_N \xi^n$ form. In the case $\kappa = 0$ the $-g_{lm}\, M^{KL} \Pi^l_L(A)\, \partial_K N^m$ term is irrelevant to the transformation of the momenta $\Pi^n_N(A)$ and the $\partial_N \xi^n$ terms in the transformation (6.111) originate from the covariant derivative terms in the $\mathcal{H}_{\text{Diff}}$ constraints.

The calculation of the transformation of the modified momenta $\mathcal{P}^n_N(A)$ can proceed in analogy to the calculation outlined in section 5.3.2 and we need to be aware of the Poisson non-commutativity of these variables. What we find is that the transformation of the $\mathcal{P}^n_N(A)$ can be expressed as in equation (6.112).

$$\begin{aligned}
\{\mathcal{P}^n_N(A), \mathcal{H}_{\mathrm{Diff}}[\xi]\} = &+ \xi^k\, \mathcal{D}_k \mathcal{P}^n_N(A) - \mathcal{D}_k\xi^n\, \mathcal{P}^k_N(A) + \mathcal{D}_k\xi^k\, \mathcal{P}^n_N(A) \\
&+ \xi^n\, (\mathcal{H}_{\mathrm{GD}})_N + \frac{1}{2}\, d_{NKL}\, \xi^k\, A^K_k\, (\mathcal{H}_{S1})^{nL} \\
&+ \partial_N \xi^k\, \Pi^n{}_a(e)\, e_k{}^a - \frac{1}{3}\, \partial_N \xi^n\, \Pi^k{}_a(e)\, e_k{}^a \\
&- \frac{1}{3}\, \partial_N \xi^n\, \Pi^{KL}(M)\, M_{KL} - \partial_K \xi^n\, \Pi^{KL}(M)\, M_{LN} \\
&+ 10\, d^{RSM}\, d_{KNR}\, \partial_S \xi^n\, \Pi^{KL}(M)\, M_{LM} \\
&+ 10\, d^{RSM}\, d_{KNR}\, \partial_S \xi^{[m}\, \mathcal{P}^{n]}_M(A)\, A^K_m \\
&- \frac{9}{2}\, \kappa\, \epsilon^{lmqn}\, d_{NQT}\, \partial_K A^Q_m\, A^K_q\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&- \frac{3}{2}\, \kappa\, \epsilon^{lmqn}\, d_{NQT}\, \partial_K A^K_q\, A^Q_m\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&+ 3\, \kappa\, \epsilon^{lmnr}\, d_{MQT}\, \partial_N A^M_r\, A^Q_m\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&+ \frac{30}{4}\, \kappa\, \epsilon^{lmqn}\, d_{MQT}\, d^{MRK}\, d_{RSN}\, \partial_K A^Q_m\, A^S_q\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&+ \frac{45}{2}\, \kappa\, \epsilon^{lmqn}\, d_{MQT}\, d^{MRK}\, d_{RSN}\, \partial_K A^S_q\, A^Q_m\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&- 15\, \kappa\, \epsilon^{lnqr}\, d_{MNT}\, d^{MRK}\, d_{RSL}\, \partial_K A^L_r\, A^S_q\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&+ 6\, \kappa\, \epsilon^{lmqn}\, d_{MNT}\, \partial_q A^M_m\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&- 30\, \kappa\, \epsilon^{lmqn}\, d^{MQR}\, d_{QNT}\, \partial_R B_{mqM}\, \partial_W \xi^k\, g_{lk}\, M^{TW} \\
&+ \kappa\, \Gamma(A,B)_N\, \xi^n
\end{aligned} \tag{6.112}$$

In the calculation of (6.112) we have to repeatedly apply the Schouten identity (equation (A.2) in appendix A) in order to move the correct index onto the gauge parameters. The two $\mathcal{P}^n_N(A)$ dependent terms in $\mathcal{H}_{\mathrm{Diff}}$ generate many topological contributions and most of these terms end up in the $\mathcal{H}_{\mathrm{S1}}$ constraints or in the complicated topological term (6.68), which is contained in the $\mathcal{H}_{\mathrm{GD}}$ constraints. What is perhaps most surprising about the calculation of (6.112) is that the rewriting (6.49) of the purely two-form dependent terms has to be used in order to match the term found in $\mathcal{H}_{\mathrm{GD}}$ and this calculation requires the application of the section condition (3.72). As far as the transformations presented in this thesis are concerned this is the only calculation outside of the canonical constraint algebra where the section condition is required in the canonical formulation of ExFT. We continue to discuss this point in chapter 7.
In the transformation (6.112) we furthermore find many topological $\partial_N \xi^n$ terms for which we do not seem to be able to find a simpler form. And finally we have written $\kappa\, \Gamma(A,B)_N\, \xi^n$ in (6.112) to capture a rather large number of topological contributions that are left over. Most of these terms that are contained in the $\Gamma(A,B)$ depend only on the one-forms and few depend on both the one- and two-forms. However there are no terms inside $\Gamma(A,B)$ that purely depend on the two-forms. We should furthermore note that these terms are not of the $\partial_N \xi^n$ form and the interpretation of these terms is not clear at this point — they might either cancel in a non-trivial way $\Gamma(A,B) = 0$ or they might form some more complicated transformation terms.

Because the two-form kinetic term in the ExFT Lagrangian is linear in the time derivative we argued in section 6.9.2 that the two-forms do not transform non-trivially under any of the secondary constraints in the Hamiltonian because they cannot contain any two-form momenta. Moreover we showed in section 6.9.2 that if one were allowed to make use of the primary constraints $\mathcal{H}_{\mathrm{P2}}$ one could rewrite the two-form term in the $\mathcal{H}_{\mathrm{GD}}$ constraints to insert the canonical momenta $\Pi^{klM}(B)$ and generate the generalised Lie derivative of the two-forms. This trick only works because of the structure of the two-form term in the $\mathcal{H}_{\mathrm{GD}}$ constraints, the same procedure does not work for the other secondary constraints, e.g. in the diffeomorphism constraints $\mathcal{H}_{\mathrm{Diff}}$ we cannot use the primary constraints at all because the two-forms only appear inside of the $\mathcal{F}_{mn}^{M}$ and $\mathcal{P}_{N}^{n}(A)$, which do not have the right structure. Even if we were allowed to use the primary constraints it would thus not be possible to make the topological two-forms transform under external diffeomorphisms canonically. This is consistent with the results of the canonical analysis of the model two-form theory in section 4.4. Canonically this should moreover be expected for any fields whose only kinetic term is located in the topological term, which by definition does not depend on the metric $G_{\mu\nu}$ and hence not on the shift vector N^n, which is the Lagrange multiplier of the diffeomorphism constraints. Thus the two-forms cannot "see" the external diffeomorphisms canonically because their kinetic term does not couple to the shift vector and their transformation (6.113) vanishes.

$$\{B, \mathcal{H}_{\mathrm{Diff}}[\xi]\} = 0 \tag{6.113}$$

The same argument applies to the transformation of the conjugated two-form momenta $\Pi^{pvS}(B)$ and we do not find any Lie derivative terms in this transformation. The transformation of the momenta $\Pi^{pvS}(B)$ is nonetheless non-vanishing because of the contributions from the two-form dependent $\mathcal{F}_{mn}^{M}$ and $\mathcal{P}_{N}^{n}(A)$ terms in the $\mathcal{H}_{\mathrm{Diff}}$ constraints. The transformation is given explicitly by (6.114).

$$\begin{aligned}\{\Pi^{pvS}(B), \mathcal{H}_{\mathrm{Diff}}[\xi]\} =& + 20\, d^{TKS}\, \partial_K \left(\xi^{[p}\, \mathcal{P}_T^{v]}(A)\right) \\ &+ 30\kappa\, \epsilon^{lpvr} d^{SNR} d_{NKT}\, \partial_R \left(\partial_L \xi^k\, g_{kl}\, M^{LT}\, A_r^K\right) \\ &- 30\kappa\, \epsilon^{lpvr} d^{SNR} d_{NKT} \partial_R \left(\xi^k\, \mathcal{F}_{kl}^{T}\, A_r^K\right)\end{aligned} \tag{6.114}$$

The expression of the transformation (6.114) is moreover equivalent to the consistency condition (6.89) when we replace the parameters by $\xi^n = N^n$. We continue the discussion of this point in section 6.9.5 when discussing the two-form constraints and the tensor gauge transformations.

The origin and sign of the term $-g_{lm}\, M^{KL} \mathcal{P}_L^l(A)\, \partial_K N^m$ in the Hamiltonian

The only diffeomorphism transformation that contains a $\partial_N \xi^n$ term which we can compare to an analogous Lagrangian gauge transformation is the transformation (6.110) of the one-forms, which we can compare to the transformation (3.112). From this comparison we can see that the sign of the $\partial_N \xi^n$ term in the canonical transformation differs from the sign of this term in the Lagrangian transformation, which is unexpected. What is clear is that the minus sign in the transformation (6.110) is a direct consequence of the sign of the $-g_{lm}\, M^{KL} \mathcal{P}_L^l(A)\, \partial_K N^m$ term in the diffeomorphism constraints of the ExFT Hamiltonian. In the remainder of this section we want to examine and explain the origin of this term in detail.

We begin by looking at the ADM decomposition of the Einstein-Hilbert improvement term (6.9) where we find the term $+\frac{e}{N}\, \mathcal{F}_{tn}^{M}\, \partial_M N^n$. This term can be found by taking

the $\mu = t$, $\nu = n, \alpha = 0$, $\beta = b, \rho = \{t, r\}$ parts of the five-dimensional indices and then making use of the identity $\partial_M(e_b{}^n\, e_r{}^b) = 0$ to unify the $\rho = \{t, r\}$ contributions. In the computation of the canonical momenta $\Pi^l_T(A)$ (6.20) the $+\frac{e}{N}\,\mathcal{F}^M_{tn}\,\partial_M N^n$ term in the Lagrangian leads to the contribution $+\frac{e}{N}\partial_T N^l$ to the $\Pi^l_T(A)$ and consequently to the modified momenta $\mathcal{P}^l_T(A)$. Due to the linearity in the time derivative, the Lagrangian term $+\frac{e}{N}\,\mathcal{F}^M_{tn}\,\partial_M N^n$ cancels in the step from (6.54) to (6.55) in the Legendre transformation against the Legendre transformation term of the one-forms $\dot{A}^N_n \cdot \Pi^n_N(A^N_n)$. What we then found, at the point (6.55) of the Legendre transformation, was that only the Yang-Mills term, which in particular contains the term $+\frac{e}{2N}\,g^{sn}\,M_{MN}\,\dot{A}^M_s\,\dot{A}^N_n$, contains time derivatives. Many terms are then generated by inserting the expression (6.57) for $\dot{A}^N_n$ into this $+\frac{e}{2N}\,g^{sn}\,M_{MN}\,\dot{A}^M_s\,\dot{A}^N_n$ term in (6.55). And focusing on the terms where one $\dot{A} \sim \mathcal{P}(A)$ and the other $\dot{A} \sim -\frac{e}{N}\partial_T N^l$ we then arrive at the term $-g_{lm}\,M^{KL}\mathcal{P}^l_L(A)\,\partial_K N^m$ in the Hamiltonian. Hence the minus sign of this term originates from the inversion of the modified momenta $\dot{A}(\mathcal{P})$ (6.57).
When the $-g_{lm}\,M^{KL}\mathcal{P}^l_L(A)\,\partial_K N^m$ term acts on the one-forms via the Poisson bracket, as in the transformation (6.110), we immediately find the contribution (6.115) to the transformation.

$$\{A^N_n, -g_{lm}\,M^{ML}\,\mathcal{P}^l_L(A)\,\partial_M N^m\} = -g_{mn}\,M^{MN}\,\partial_M N^m \tag{6.115}$$

Regarding the Lagrangian gauge transformation in section 3.5.3, we explained the origin of the analogous term in the transformation (3.112) as originating from a compensating Lorentz transformation when considering the Kaluza-Klein-like rewriting of eleven-dimensional supergravity (see also references [25, 103]). What we found in section 3.5.3 was that there does not seem to be any choice in the derivation of this sign coming from the gauge transformations of eleven-dimensional supergravity.
Moreover the difference in sign between the Hamiltonian and Lagrangian formalism could possibly be explained by a diverging convention, but the conventions chosen in this work and in particular the signature of the Minkowski metric, seem to agree with the conventions that are used in the references [3, 25].

Looking at the transformation (6.110) from a purely canonical perspective it is not immediately clear that the sign in (6.110) is problematic, although it remains to be examined whether this sign might affect the closure of the canonical constraint algebra.

With all the above factors taken into consideration we do not have an explanation for the difference in the sign of the $\partial_N \xi^n$ term in the transformation (6.110) when compared to the analogous term in the Lagrangian gauge transformation (3.112).

6.9.4 Time evolution

Acting on the canonical coordinates with the Hamilton constraint $\mathcal{H}_{\text{Ham}}$ (6.82) generates the time evolution. The time evolution that is generated by the Hamilton constraint on the spatial vielbein (6.116), the scalar fields (6.117) and the one-forms (6.118) is in form identical to the time evolution that we have seen in $E_{6(6)}$ invariant five-dimensional supergravity in section 5.3.2.

$$\{e_{na}, \mathcal{H}_{\text{Ham}}[\phi]\} = +\frac{\phi}{2e}\, g_{mn}\, \Pi^m{}_a(e) - \frac{\phi}{6e}\, \Pi(e)\, e_{na} \tag{6.116}$$

$$\{M_{MN}, \mathcal{H}_{\text{Ham}}[\phi]\} = +\frac{6}{e}\phi\, \Pi^{QP}(x)\, M_{MQ}\, M_{NP} \tag{6.117}$$

$$\{A_n^N, \mathcal{H}_{\text{Ham}}[\phi]\} = +\frac{\phi}{e}\, g_{nl}\, M^{NL}\, \mathcal{P}^l_L(A) \tag{6.118}$$

The form of the time evolutions of these fields is identical in both theories because the canonical momenta terms in the Hamilton constraint are of the same form in both theories. We can see the transformation (6.118) as another argument in favour of the modified momenta $\mathcal{P}^n_N(A)$ because the time evolution of the one-forms is most concisely written in terms of the modified variables.

In contrast to the time evolution of the fields the time evolution of the canonical momenta is significantly more complicated in the ExFT when compared to that of $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity. Some of the five-dimensional transformations are already relatively complicated, but much of the added complexity comes from the scalar potential, the topological contributions and the covariant external derivatives.

In the canonical formulation of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity the canonical time evolution of the vielbein momenta $\Pi^n{}_a(e)$ is given by (5.48), which contains the spatial Einstein equation in vielbein form and contributions from all other terms in the Hamilton constraint because the metric couples to all fields. The analogous covariantised time evolution in the ExFT contains a number of additional terms, mostly due to the scalar potential (6.67) which does not exist in five-dimensional $\mathrm{E}_{6(6)}$ invariant supergravity. The canonical time evolution of the vielbein momenta in ExFT is given by (6.119), the covariant derivative ∇_n contains only the Levi-Civita connection.

$$
\begin{aligned}
\{\Pi^n{}_a(e), \mathcal{H}_{\mathrm{Ham}}[\phi]\} = &+ \frac{\phi}{4e}\, \Pi_{bc}(e)\, \Pi_{bc}(e)\, e_a{}^n - \frac{\phi}{2e}\Pi^k{}_b(e)\, \Pi^n{}_b(e)\, e_{ka} \\
&- \frac{\phi}{12e}\, \Pi^2(e)\, e_a{}^n + \frac{\phi}{6e}\, \Pi(e)\, \Pi^n{}_a(e) \\
&- 2\phi e \left(\hat{R}^{nk}\, e_{ka} - \frac{1}{2}\hat{R}\, e_a{}^n\right) \\
&+ 2e \left(\nabla_a \nabla^n \phi - \nabla^k \nabla_k \phi\, e_a{}^n\right) \\
&+ \frac{3\phi}{2e}\Pi^{MN}(M)\, \Pi^{RS}(M)\, M_{MR}\, M_{NS}\, e_a{}^n \\
&+ \frac{\phi e}{24}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{kl}\, e_a{}^n - \frac{\phi e}{12}\partial_k M_{MN}\, \partial_l M^{MN}\, g^{ln}\, e_a{}^k \\
&- \frac{\phi e}{4} M_{MN}\, \mathcal{F}^M_{rs}\, \mathcal{F}^N_{km}\, g^{rk}\, g^{sm}\, e_a{}^n + \phi e\, M_{MN}\, \mathcal{F}^M_{rs}\, \mathcal{F}^N_{km}\, g^{rk}\, g^{mn}\, e_a{}^s \\
&+ \frac{\phi}{2e} M^{KL}\, \mathcal{P}^l_L(A)\, \mathcal{P}^k_K(A)\, g_{lk}\, e_a{}^n - \frac{\phi}{e} M^{KL}\, \mathcal{P}^n_K(A)\, \mathcal{P}^l_L(A)\, e_{la} \\
&+ \frac{e\phi}{4} M^{MN}\, \partial_M g^{kl}\, \partial_N g_{kl}\, e_a{}^n + \partial_M \left(\frac{e\phi}{2} M^{MN}\, \partial_N g_{kl}\right) g^{kn}\, e_a{}^l \\
&- \partial_N \left(\frac{e\phi}{2} M^{MN} \partial_M g^{mn}\right) e_{ma} \\
&- \frac{\phi}{e}\, M^{MN}\, \partial_M e\, \partial_N e\, e_a{}^n - 2\,\partial_N \left(\frac{\phi}{e} M^{MN}\, \partial_M e\right)\, e\, e_a{}^n \\
&+ \frac{e\,\phi}{24} M^{MN} \partial_M M^{KL} \partial_N M_{KL}\, e_a{}^n - \frac{e\,\phi}{2} M^{MN} \partial_M M^{KL} \partial_L M_{NK}\, e_a{}^n \\
&- e\phi \partial_M \partial_N M^{MN} e_a{}^n - 2e\partial_M \partial_N \phi\, M^{MN} e_a{}^n - 2e\partial_N \phi\, \partial_M M^{MN} e_a{}^n \\
&- 2\, e\, \phi\, \mathcal{F}^M_{mk}\, g^{mn}\, \partial_M e^k_a - \partial_M \left(e\, \phi\, \mathcal{F}^M_{mk}\, g^{mn}\, e^k_a\right) \\
&- e\, \phi\, \mathcal{F}^M_{mk}\, \partial_M e^b_r \left(e^n_b\, e^k_a\, g^{mr} + e^k_b\, e^m_a\, g^{rn} + e^k_b\, e^r_a\, g^{mn}\right)
\end{aligned}
\tag{6.119}
$$

In $E_{6(6)}$ invariant five-dimensional supergravity the canonical time evolution of the scalar momenta $\Pi^{KL}(M)$ (5.50) is relatively simple. In (5.50) we can see the contributions from the scalar kinetic terms and the transformation is only slightly complicated by the use of the scalar fields M_{MN} as a generalised metric to contract the $E_{6(6)}$ indices in the Yang-Mills term. In the analogous transformation in ExFT these terms also exist, but there are numerous additional contributions coming from the covariant derivatives in the scalar kinetic terms and in particular also from the scalar potential (6.67). The canonical time evolution of the scalar momenta in ExFT is given by (6.120).

$$
\begin{aligned}
&\{\Pi^{KL}(M), \mathcal{H}_{\text{Ham}}[\phi]\} = -\frac{6\phi}{e}\ \Pi^{PK}(M)\ \Pi^{LR}(M)\ M_{PR} \qquad (6.120)\\
&- \partial_l \left(\frac{\phi e}{6}\ g^{kl}\ \partial_k M^{KL}\right) - \frac{\phi e}{6}\ g^{kl}\ \partial_k M_{MN}\ \partial_l M^{KM}\ M^{LN}\\
&- \frac{\phi e}{2}\ g^{rm}\ g^{sn}\ \mathcal{F}^K_{rs}\ \mathcal{F}^L_{mn} + \frac{\phi}{e} g_{lm}\ \mathcal{P}^l_M(A)\ \mathcal{P}^m_N(A)\ M^{KM}\ M^{LN}\\
&- \partial_n \left(\frac{e\,\phi}{12}\, g^{mn}\, \mathbb{L}_{A_m} M_{MN}\right) M^{KM}\, M^{LN} + \partial_n \left(\frac{e\,\phi}{12}\, g^{mn}\, \mathbb{L}_{A_m} M^{KL}\right)\\
&+ \partial_R \left(\frac{e\,\phi}{12}\, g^{mn}\, \mathcal{D}_n M^{KL}\, A^R_m\right) - \frac{e\,\phi}{18} g^{mn}\, \mathcal{D}_n M^{KL}\, \partial_R A^R_m\\
&- \frac{e\,\phi}{6} g^{mn}\, \mathcal{D}_n M^{M(K} \partial_M A^{L)}_m + \frac{5\,e\,\phi}{3} g^{mn}\, \mathcal{D}_n M^{M(K}\, d^{L)TX}\, d_{UXM}\, \partial_T A^U_m\\
&- \partial_R \left(\frac{e\,\phi}{12} g^{mn}\, \mathcal{D}_n M_{MN}\, A^R_m\right) M^{KM}\, M^{LN} - \frac{e\,\phi}{18} g^{mn}\, \mathcal{D}_n M_{MN}\, \partial_R A^R_m\, M^{KM}\, M^{LN}\\
&- \frac{e\,\phi}{6} g^{mn}\, \mathcal{D}_n M_{MN}\, \partial_R A^M_m\, M^{N(K}\, M^{L)R}\\
&+ \frac{5\,e\,\phi}{3} g^{mn}\, \mathcal{D}_n M_{MN}\, d^{XTM}\, d_{RUX}\, \partial_T A^U_m\, M^{N(K}\, M^{L)R}\\
&- \frac{e\,\phi}{2} \partial_M g^{mn}\, \partial_N g_{mn}\, M^{M(K}\, M^{L)N} - \frac{2\phi}{e} \partial_M e\, \partial_N e\, M^{KM}\, M^{LN}\\
&+ 4\phi\, \partial_M \partial_N e\, M^{KM}\, M^{LN} + 2\partial_N \partial_M (e\phi)\, M^{KM}\, M^{LN} - 4\partial_M (\phi \partial_N e)\, M^{M(K}\, M^{L)N}\\
&- \frac{e\,\phi}{12} \partial_M M^{RS}\, \partial_N M_{RS}\, M^{M(K}\, M^{L)N} + \partial_M \left(\frac{e\,\phi}{12} M^{MN}\, \partial_N M_{RS}\right) M^{KR}\, M^{LS}\\
&- \partial_N \left(\frac{e\,\phi}{12} \partial_M M^{KL}\, M^{MN}\right) + \partial_S \left(e\,\phi\, M^{M(K}\, \partial_M M^{L)S}\right)\\
&+ e\,\phi\, \partial_M M^{RS}\, \partial_S M_{NR}\, M^{M(K}\, M^{L)N} - \partial_M \left(e\,\phi\, M^{MN}\, \partial_S M_{NR}\right) M^{R(K}\, M^{L)S}
\end{aligned}
$$

It seems likely that there exists a slightly simpler and more covariant form of expressing the terms in (6.120) that are coming from the scalar kinetic term, but there seems to be little hope of significantly simplifying the scalar potential contributions.

In ungauged $E_{6(6)}$ invariant five-dimensional supergravity the canonical time evolution of the modified one-form momenta $P^k_K(A)$, which is given by (5.52), is quite simple and only consists of two terms. The first term in (5.52) is the contribution from the abelian Maxwell-like kinetic term and the second term is a topological contribution that originates from the Poisson non-commutativity of the modified momenta $P^k_K(A)$, but due to the simple topological term in five-dimensions this does not complicate the transformation much. In the ExFT we have used the one-forms A^M_μ to gauge the generalised diffeomorphism symmetry and as a consequence the canonical time evolution of their conjugate modified momenta $\mathcal{P}^k_K(A)$ becomes very complicated. These complications originate in part from the more complicated covariant non-abelian field strength terms and from the covariant derivatives of other fields, but in particular there are numerous topological contributions from the Poisson non-commutativity of the modified momenta $\mathcal{P}^k_K(A)$ with the $\mathcal{P}^2$ term of the Hamilton constraint. Due to the complexity of the calculation we do not give the full transformation of $\mathcal{P}^k_K(A)$ here, but the canonical time evolution of $\Pi^k_K(A)$, for the case $\kappa = 0$ without the topological contributions, can be expressed as (6.121).

$$
\begin{aligned}
\{\Pi^k_K(A), \mathcal{H}^{\kappa=0}_{\text{Ham}}[\phi]\} = &+ \partial_m \Big(e\,\phi\; M_{MK}\; g^{rm}\; g^{ks}\; \mathcal{F}^M_{rs}\Big) \\
&- \partial_R \Big(e\,\phi\; M_{MK}\; g^{rm}\; g^{ks}\; \mathcal{F}^M_{rs}\; A^R_m\Big) \\
&+ e\,\phi\; M_{MN}\; g^{rk}\; g^{sn}\; \mathcal{F}^M_{rs}\, \partial_K A^N_n \\
&- 5\, d^{NRS}\, d_{RKL}\, e\,\phi\; M_{MN}\; g^{rk}\; g^{sn}\; \mathcal{F}^M_{rs}\, \partial_S A^L_n \\
&+ 5\, d^{NRS}\, d_{RKL}\, \partial_S \Big(e\,\phi\; M_{MN}\; g^{rm}\; g^{ks}\; \mathcal{F}^M_{rs}\; A^L_m\Big) \\
&- \frac{e\phi}{12}\, g^{kn}\, \mathcal{D}_n M^{MN}\, \partial_K M_{MN} \\
&+ \mathbb{P}^R{}_M{}^L{}_K\, \partial_L \Big(e\,\phi\, g^{kn}\, \mathcal{D}_n M^{MN}\, M_{RN}\Big) \\
&- 2\, \partial_m \Big(e\,\phi\, e_a{}^{[m}\, e_b{}^{k]} (e^{ar} \partial_K e_r{}^b)\Big) \\
&+ 2\, \partial_R \Big(e\,\phi\, A^R_m\, e_a{}^{[m}\, e_b{}^{k]} (e^{ar} \partial_K e_r{}^b)\Big) \\
&- 2\, e\,\phi\, \partial_K A^M_n\, e_a{}^{[k}\, e_b{}^{n]} (e^{ar} \partial_M e_r{}^b) \\
&+ 10\, d^{MRS}\, d_{RLK}\, e\,\phi\, \partial_S A^L_n\, e_a{}^{[k}\, e_b{}^{n]} (e^{ar} \partial_M e_r{}^b) \\
&- 10\, d^{MRS}\, d_{RLK}\, \partial_S \Big(e\,\phi\, A^L_m\, e_a{}^{[m}\, e_b{}^{k]} (e^{ar} \partial_K e_r{}^b)\Big)
\end{aligned}
\tag{6.121}
$$

In the first line of (6.121) we can recognise the covariantised version of the transformation (5.52), the topological term is missing here because the expression above is given at $\kappa = 0$. The remainder of the terms in the transformation (6.121) originate from the covariantised one-form field strengths in the Einstein-Hilbert term and generalised Yang-Mill term, as well as from the covariant external derivatives in the scalar kinetic term.

In the previous sections we have already explained for the external and internal diffeomorphism constraints that, due to the Lagrangian two-form kinetic term being topological and linear in the time derivative, the two-forms cannot transform canonically under these secondary constraints. The same argument applies to the Hamilton constraint and we find that the canonical time evolution of the two-forms that is generated by the Hamilton constraint (6.122) vanishes.

$$
\{B_{klM}, \mathcal{H}_{\text{Ham}}[\phi]\} = 0 \tag{6.122}
$$

Equation (6.122) moreover implies that the overall canonical time evolution of the two-forms consists entirely of the transformations generated by other canonical constraints that are not vanishing. We have already confirmed in the topological model theory in section 4.4 that there are no propagating degrees of freedom for this two-form kinetic term and this fact should not be changed by the topological and Stückelberg coupling to the other fields. In contrast to the canonical time evolution of the propagating fields, which is given in terms of their canonical momenta (cf. (6.116), (6.117) and (6.118)) we should therefore not expect the canonical time evolution of the two-forms to be given in terms of $\Pi^{klM}(B)$. Furthermore the construction of a $(\Pi(B))^2$ term in the Hamilton constraint is not possible, even if we make use of the primary constraints $\mathcal{H}_{\text{P2}}$, because the Stückelberg coupling in the one-form field strength does not have the same structure as the $\mathcal{H}_{\text{P2}}$ constraints.

Because of the Stückelberg coupling in $\mathcal{F}^{M}_{mn}$ and because of the two-form term that is hidden inside the modified one-form momenta $\mathcal{P}^{m}_{M}$, the canonical two-form momenta $\Pi^{pvW}(B)$ do transform under the secondary constraints, including the Hamilton constraint. The Hamilton constraint generated the transformation (6.123) on the two-form momenta $\Pi^{pvW}(B)$.

$$\begin{aligned}\{\Pi^{pvW}(B), \mathcal{H}_{\text{Ham}}[\phi]\} = &- \partial_N \left(20\, d^{WMN} \phi\, e\, e_a{}^{[p}\, e_b{}^{v]}\, (e^{ra}\, \partial_M e_r{}^b)\right) \\ &+ \partial_R \left(100\, \phi\, e\, d^{MKL}\, d^{NRW}\, M_{MN}\, g^{rp}\, g^{sv}\, \partial_K B_{rsL}\right) \\ &- \partial_K \left(\frac{30\kappa\phi}{e}\, g_{mn}\, M^{MN}\, \mathcal{P}^n_N\, A^S_q\, \epsilon^{tmpvq}\, d^{RKW}\, d_{MRS}\right)\end{aligned} \tag{6.123}$$

The transformation (6.123) should not be interpreted as a normal time evolution due to the above argument and due to the results from the topological model theory in section 4.4. Moreover the expression (6.123) corresponds to the consistency condition (6.88) when the transformation parameter is replaced by the lapse function $\phi = N$.

6.9.5 $B_{\mu\nu M}$ tensor gauge transformations

In this section the transformations that are generated by the constraints $\mathcal{H}_{\text{P1}}$ (6.37), $\mathcal{H}_{\text{P2}}$ (6.38) and $\mathcal{H}_{\text{S1}}$ (6.85) are discussed and we can identify some of these transformations with the tensor gauge transformations of the two-forms. Furthermore we comment on the consistency conditions (6.88), (6.89), (6.90) and (6.91) in this section, which can be seen as being analogous to the $\mathcal{H}_{\text{S2}}$ constraints in the topological model theory from section 4.4.

Because the form of the primary constraints $\mathcal{H}_{\text{P1}}$ and $\mathcal{H}_{\text{P2}}$ is identical to that of the eponymous constraints from the model theory of section 4.4 they generate the same transformations canonically. The primary constraints $\mathcal{H}_{\text{P1}}$ (6.37), which are of shift type, only generate the shift transformations (6.124) on the time components of the two-forms B_{tmN}, which are conjugate to them.

$$\{B_{tmN}, \mathcal{H}_{\text{P1}}[\lambda]\} = 2\, \lambda_{mN} \tag{6.124}$$

Likewise the primary constraints $\mathcal{H}_{\text{P2}}$ generate the shift transformations (6.125) on the spatial two-form components B_{mnS}.

$$\{B_{mnS}, \mathcal{H}_{\text{P2}}[\rho]\} = 2\, \rho_{mnS} \tag{6.125}$$

$$\{\Pi^{mnS}(B), \mathcal{H}_{\text{P2}}[\rho]\} = 30\kappa\, \epsilon^{tmnkl}\, d^{SRN}\, \partial_R \rho_{klN} \tag{6.126}$$

$$\{\mathcal{P}^m_M(A), \mathcal{H}_{\text{P2}}[\rho]\} = -30\, \kappa\, \epsilon^{tklmn}\, d^{KST}\, d_{TNM}\, A^N_n\, \partial_S \rho_{klK} \tag{6.127}$$

As has been illustrated in equation (4.82), the general shift transformations (6.124) and (6.125) in particular include the more specific tensor gauge transformations $\Xi_{\mu M}$ (3.107), as well as the restricted $\mathcal{O}_{\mu\nu M}$ shift transformations (3.107). Although in principle there should be a way of explicitly bringing these canonical shift transformations into the usual Lagrangian form, which was also discussed in section 4.4, it is at this point not clear how this can be achieved for the tensor gauge transformations. The $\mathcal{H}_{\text{P2}}$ constraints generate the transformation (6.126) on the momenta $\Pi^{mnS}(B)$, which is identical to the transformation (4.79) of the model two-form theory.
The modified momenta $\mathcal{P}^m_M(A)$ transform as (6.127), because of the topological two-form term that is hidden inside them. We should however not think of this as a new

transformation, because (6.127) is induced by the transformation (6.125) in the composite expression $\mathcal{P}^m_M(A)$ which is not a fundamental canonical coordinate.

The secondary constraints $\mathcal{H}_{\mathrm{S1}}$ generate the transformation (6.128) on the one-forms A_n^N.

$$\{A_n^N, \mathcal{H}_{\mathrm{S1}}[\Xi]\} = -10\, d^{LMN}\, \partial_L \Xi_{nM} \tag{6.128}$$

Moreover we can identify the transformation (6.128) precisely as the tensor gauge transformation of A_n^N, which is induced by the Stückelberg coupling, by comparing the canonical transformation (6.128) to the Lagrangian $\Xi_{\mu M}$ transformation in equation (3.106). The one-form momentum term in (6.92) originates from the Stückelberg coupling and is thus not dependent on the topological term, which means that the transformation (6.128) of A_n^N exists even in the case $\kappa = 0$.

By contrast the canonical momenta $\Pi^n_N(A)$ do not transform (6.129) under $\mathcal{H}_{\mathrm{S1}}$ in the $\kappa = 0$ case.

$$\{\Pi^n_N(A), \mathcal{H}^{\kappa=0}_{\mathrm{S1}}[\Xi]\} = 0 \tag{6.129}$$

Due to the large number of terms originating from the Poisson non-commutativity of the modified momenta $\mathcal{P}^n_N$ and due to the covariantisation terms in $\mathcal{H}_{klmN}$ the explicit transformation of the modified momenta $\mathcal{P}^n_N$ generated by $\mathcal{H}_{\mathrm{S1}}$ is very complicated and remains to be calculated.

In agreement with the results of section 4.4 we find that the transformation (6.130) of the two-forms, generated by the secondary constraints $\mathcal{H}_{\mathrm{S1}}$, vanishes. The transformation generated by the $\mathcal{H}_{\mathrm{S1}}$ constraints on the two-form momenta $\Pi^{qsR}(B)$ is non-vanishing and given by (6.131), which we can expand as (6.132).

$$\begin{aligned}
\{B_{qsR}, \mathcal{H}_{\mathrm{S1}}[\Xi]\} &= 0 && (6.130)\\
\{\Pi^{qsR}(B), \mathcal{H}_{\mathrm{S1}}[\Xi]\} &= +\, 300\, \kappa\, \epsilon^{lqsr}\, d^{MKL}\, d^{SNR}\, d_{QSL}\, \partial_N \left(\partial_K \Xi_{lM}\, A_r^Q\right) && (6.131)\\
&\quad +\, 60\, \kappa\, \epsilon^{lmqs}\, d^{MKR}\, \mathcal{D}_m \partial_K \Xi_{lM}\\
&= +\, 300\, \kappa\, \epsilon^{lqsr}\, d^{MKL}\, d^{SNR}\, d_{QSL}\, \partial_N \left(\partial_K \Xi_{lM}\, A_r^Q\right) && (6.132)\\
&\quad +\, 60\, \kappa\, \epsilon^{lmqs}\, d^{MKR}\, \partial_m \partial_K \Xi_{lM}\\
&\quad -\, 60\, \kappa\, \epsilon^{lqsr}\, d^{MKR}\, A_r^N\, \partial_N \partial_K \Xi_{lM}\\
&\quad +\, 60\, \kappa\, \epsilon^{lmqs}\, d^{MKN}\, \partial_K \Xi_{lM}\, \partial_N A_m^R\\
&\quad -\, 600\, \kappa\, \epsilon^{lmqs}\, d^{MKL}\, d^{RPS}\, d_{LTP}\, \partial_s A_m^T\, \partial_K \Xi_{lM}
\end{aligned}$$

In the transformation (6.131) we can recognise the covariantised version of (4.80) plus an additional one-form dependent term. In the expanded expression (6.132) we can furthermore see that, due to the different structures of the terms, it is not possible to simplify this expression in any meaningful way.

The expression (6.132) is equivalent to the consistency condition (6.91) when the transformation parameter is replaced by the time components of the two-forms $\Xi_{nM} = B_{tnM}$. In this case the expression is in direct analogy to the $\mathcal{H}_{\mathrm{S2}}$ constraints (4.72) from the two-form model in section 4.4. Although the $\mathcal{H}_{\mathrm{S2}}$ constraints consisted only of the second term in (6.132) for the model theory. With the expression (6.91) we now have the explicit form of each of the consistency conditions (6.88), (6.89), (6.90) and (6.91), all of which follow from the conditions (6.87). In analogy to the (second class) constraints $\mathcal{H}_{\mathrm{S2}}$ from the model theory in section 4.4, these consistency conditions are all non-vanishing and independent from the other constraints, which means that we should think of them as canonical constraints. These constraints originate from the combination of the topological term (3.135) with the Stückelberg coupling in the one-form field strength (3.90).

What makes these constraints somewhat unusual is their dependence on the Lagrange multipliers N, N^n, A_t^M and B_{tnN}. We have already seen this structure in the model theory in section 4.4 where the $\mathcal{H}_{\mathrm{S2}}$ constraints depend on the Lagrange multipliers B_{tnN}. Moreover we have found that this circumstance leads to the constraint algebra relation (4.83), which means that $\mathcal{H}_{\mathrm{S2}}$ and the primary constraints $\mathcal{H}_{\mathrm{P1}}$ are second class constraints in the model theory and a similar relation should hold true in ExFT. In the two-form model theory the $\mathcal{H}_{\mathrm{S2}}$ constraints are themselves of the form of the algebra relation (4.84) which is responsible for making the other constraints second class and this seems to suggest some relationship between these unusual constraints and the need for a Dirac bracket. In the ExFT it may be possible that a similar relation for the constraints (6.88), (6.89), (6.90) and (6.91) means that the shift type primary constraints, which are simultaneously the canonical momenta conjugate to the Lagrange multipliers, become second class.
A better understanding of the model theory from section 4.4 and in particular of the $\mathcal{H}_{\mathrm{S2}}$ constraints and the Dirac bracket is needed to make sense of these constraints.

6.9.6 Shifts and Lorentz transformations

In this section we briefly describe the transformations generated by the remaining primary constraints listed in section 6.4, which are either of the shift type form or the Lorentz constraints. These transformations are identical to those that appear in the canonical analysis of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity in chapter 5.

The shift type primary constraints generate the shift transformations of the conjugate canonical variables (Lagrange multipliers) that are listed below.

$$\{N, \Pi(N)[\lambda_1]\} = \lambda_1 \tag{6.133}$$

$$\{N_a, \Pi(N_b)[\lambda_2]\} = (\lambda_2)_a \tag{6.134}$$

$$\{A_t^N, \Pi(A_t^M)[\lambda_3]\} = (\lambda_3)^N \tag{6.135}$$

The spatial Lorentz transformations of the spatial vielbein and their conjugate momenta are generated by the Lorentz constraints (6.39) and can be explicitly written as follows.

$$\{e_n{}^a, L[\gamma]\} = +\, e_{nb}\ \gamma^{ba} \tag{6.136}$$

$$\{\Pi^n{}_a(e), L[\gamma]\} = +\, \Pi^n{}_c(e)\ \gamma^{cb}\ \delta_{ba} \tag{6.137}$$

These Lorentz transformations are identical to those of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity which have already been discussed in detail in chapter 5.

6.10 Canonical constraint algebra

In this section the algebra formed by the canonical constraints under the Poisson bracket is discussed. For some of the relations in the constraint algebra we can only give speculative results because not all of the transformations of the modified one-form momenta $\mathcal{P}^m_M$ have been fully computed and because some of the calculations concerning the constraint algebra are very difficult to compute.

With the exception of the Lorentz constraints, which form the Lorentz subalgebra (6.138), all of the primary constraints Poisson-commute. In section 4.4 the Poisson-commutativity of the primary two-form constraints $\mathcal{H}_{\mathrm{P1}}$ and $\mathcal{H}_{\mathrm{P2}}$ has already been verified.

Furthermore we can examine the algebra relations which concern the Lorentz constraints and the secondary constraints. In chapter 5 we found that the Lorentz constraints Poisson-commute with the Hamilton constraint in the canonical formulation of $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity. The ExFT scalar potential is Lorentz invariant because it can be expressed entirely in terms of the metric and hence we find that the Lorentz constraints Poisson-commute with the ExFT Hamilton constraint (6.139).

$$\{L[\gamma], L[\kappa]\} = L[-2\gamma^{c[a}\ \kappa^{b]c}] \tag{6.138}$$

$$\{\mathcal{H}_{\mathrm{Ham}}[\phi], L[\gamma]\} = 0 \tag{6.139}$$

$$\{\mathcal{H}_{\mathrm{Diff}}[\lambda], L[\gamma]\} = L[\lambda^m \mathcal{D}_m \gamma^{ab}] \tag{6.140}$$

$$\{\mathcal{H}_{\mathrm{GD}}[\Lambda], L[\gamma]\} = L[\mathbb{L}_\Lambda \gamma^{ab}] \tag{6.141}$$

$$\{\mathcal{H}_{\mathrm{S1}}[\Xi], L[\gamma]\} = 0 \tag{6.142}$$

The Poisson bracket of the Lorentz constraints with the external diffeomorphism constraints can be written as (6.140), which is another Lorentz constraint term but the gauge parameters are the covariantised Lie derivative of the original Lorentz parameters. Moreover the relation (6.140) is the covariantised version of the equivalent algebra relation (5.76) from the canonical analysis of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity. The contribution from the $-g_{lm}\, M^{KL}\mathcal{P}^l_L(A)\,\partial_K N^m$ term in the $\mathcal{H}_{\mathrm{Diff}}$ constraints to the relation (6.140) vanishes, irrespectively of the sign of the term, due to the antisymmetry of the Lorentz constraints.
Equation (6.141) states the Poisson bracket of the Lorentz constraints with the generalised diffeomorphism constraints and we can see that the parameters of the resulting Lorentz constraints are the generalised Lie derivative of the original parameters. The algebra relation (5.77) of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity is analogous to the relation (6.141) and vanishes, this is consistent because in the trivial solution of the ExFT section condition the generalised Lie derivative vanishes.
Due to the form of the $\mathcal{H}_{\mathrm{S1}}$ constraints they Poisson-commute with the Lorentz constraints and we find the relation (6.142).

In the above algebra relations we did not need to make use of the section condition (3.72). In the computation of the algebra relation (6.143) which concerns the generalised diffeomorphism constraints however we need to make use of the section condition many times.

$$\{\mathcal{H}_{\mathrm{GDC}}[\Lambda], \mathcal{H}_{\mathrm{GDC}}[\zeta]\} = \mathcal{H}_{\mathrm{GDC}}\,[[\Lambda, \zeta]_E] + \dots \tag{6.143}$$

Canonically the computation of the relation (6.143) is much more complicated than in the Lagrangian formalism. This is in part due to the fact that the $\mathcal{H}_{\mathrm{GD}}$ constraints do not just generate generalised diffeomorphisms but also contain some information about the tensor gauge transformations (cf. equations (6.99) and (6.102)). Furthermore it is easy to see that the $\mathcal{H}_{\mathrm{GD}}$ constraints are much complicated than the generalised Lie

derivative and due to the Poisson non-commutativity of the modified momenta $\mathcal{P}^m_M$ and due to the Hamiltonian topological term inside of the $\mathcal{H}_{\text{GD}}$ numerous terms are generated in the computation of (6.143). The constraint algebra relation of the generalised diffeomorphism constraints (6.143) has been verified only for the $\kappa = 0$ case. In this computation there are additional terms which one might be able to rearrange into further constraints that are possibly related to the tensor gauge transformations, but the details of this remain to be computed. In the computation of (6.143) one needs to make use of the cubic d-symbol relations (3.67) and (3.68) repeatedly in order to move the $\mathrm{E}_{6(6)}$ indices between objects.
The fact that the canonical constraints act in the Poisson brackets from the right onto the fields explains the seeming difference in sign between the canonical relation (3.127) and the relation (6.143) of the Lagrangian gauge transformation. We can moreover verify this explicitly by making use of the relation $\delta_{\mathcal{H}_{\text{GDC}}[\Lambda]} = \{\,\cdot\,, \mathcal{H}_{\text{GDC}}[\Lambda]\}$ to translate the Lagrangian relation (3.126) into the canonical formalism and then making use of the Jacobi identity and the antisymmetry of the Poisson bracket.

To be consistent the canonical constraint algebra of the $\mathrm{E}_{6(6)}$ ExFT has to reduce to the constraint algebra of the canonical formulation of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity, which was comprehensively calculated in section 5.3.3, when applying the trivial solution of the section condition. We can thus combine the canonical results from section 5.3.3 with the Lagrangian gauge algebra, which was calculated in reference [36] and which we summarised in section 3.5, to make some conjectures about the remaining canonical constraint algebra relations. For each of the following conjectured constraint algebra relations it is possible that additional canonical constraints appear and this is particularly true for the two-form constraints which do not have any analogues in five dimensions. Due to the fact that one can always choose a different basis for the constraint algebra, the constraint algebra of the Hamiltonian formalism does not have to be identical in form to that of the gauge algebra in the Lagrangian formalism.

In analogy to the Lagrangian algebra relations (3.120) and (3.121) one might conjecture the canonical constraint algebra relations (6.144) and (6.145), which concern the external and internal diffeomorphism constraints — these relations are conjectures and have not been verified computationally.

$$\begin{aligned}\{\mathcal{H}_{\text{Diff}}[\lambda], \mathcal{H}_{\text{Diff}}[\xi]\} \overset{?}{=}\, & \mathcal{H}_{\text{Diff}}\left[\lambda^\mu\, \mathcal{D}_m \xi^n - \xi^m\, \mathcal{D}_m \lambda^n\right] \\ & + \mathcal{H}_{\text{GD}}[\cancel{\lambda^m\, \xi^n\, \mathcal{F}^M_{mn}} + M^{MN}\, g_{mn}\, (\lambda^m \partial_M \xi^n - \xi^m \partial_M \lambda^n)] + \dots\end{aligned} \tag{6.144}$$

$$\begin{aligned}\{\mathcal{H}_{\text{GD}}[\Lambda], \mathcal{H}_{\text{Diff}}[\xi]\} \overset{?}{=}\, & \mathcal{H}_{\text{Diff}}[\mathbb{L}_\Lambda \xi^n] \\ & + \mathcal{H}_{\text{S1}}[d_{MNK}\, \Lambda^K\, (\xi^m\, \mathcal{F}^N_{mn} - M^{KL}\, g_{mn}\, \partial_L \xi^n)] + \dots\end{aligned} \tag{6.145}$$

The field-strength term in the parameter of the $\mathcal{H}_{\text{GD}}$ constraints in (6.144) seems to be problematic because this term does not vanish in the trivial solution of the section condition. This however would be inconsistent with the relation (5.71) from the canonical formulation of $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity and thus it seems that this term should not appear canonically — possibly due to a different parametrisation of the algebra with respect to the Lagrangian formulation. Further uncertainty about the precise form of these relations is added by the fact that the sign of the $\partial_M \xi^m$ term from the transformation (6.110) appears in two of the above gauge parameters. On the other hand the $\mathcal{H}_{\text{Diff}}$ constraint terms on the right hand sides of both of these relations seem rather likely to be correct. Due to the involvement of the generalised diffeomorphism constraints it seems probable that the section condition may play a role in the computation of the above relations.

The constraint algebra relation $\{\mathcal{H}_{\mathrm{Ham}}[\theta], \mathcal{H}_{\mathrm{GD}}[\xi]\}$ is especially difficult to compute due to the number of modified one-form momentum terms involved. Because the analogous relation vanishes in the canonical formulation of the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity we cannot extrapolate the result to the ExFT and in the Lagrangian formulation there is no analogue to this algebra relation.
In analogy to the constraint algebra relations (5.68) and (5.69) from the canonical analysis of the five-dimensional supergravity we may finally conjecture the relations (6.146) and (6.147) for the canonical formulation of ExFT.[2]

$$\begin{aligned}\{\mathcal{H}_{\mathrm{Ham}}[\theta], \mathcal{H}_{\mathrm{Ham}}[\tau]\} &\stackrel{?}{=} \mathcal{H}_{\mathrm{Diff}}[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}] \\ &\qquad - L\,[(\theta\,\nabla_m\tau - \tau\,\nabla_m\theta)\,g^{mn}\,\omega_{nab}] + \ldots \qquad (6.146)\\ \{\mathcal{H}_{\mathrm{Diff}}[\lambda], \mathcal{H}_{\mathrm{Ham}}[\theta]\} &\stackrel{?}{=} \mathcal{H}_{\mathrm{Ham}}[\lambda^m\,\mathcal{D}_m\theta] + \mathcal{H}_{\mathrm{GD}}\left[\frac{\theta}{e}\lambda^p\,g_{pk}\,\mathcal{P}^k_L\,M^{LM}\right] + \ldots \qquad (6.147)\end{aligned}$$

Additional (two-form) constraint terms might possibly appear on the right hand sides of the relations (6.146) and (6.147).

Because there are no two-forms in the $\mathrm{E}_{6(6)}$ invariant five-dimensional supergravity we are unable to get any information about the algebra relations concerning the two-form constraints from the results of chapter 5. Nonetheless we should probably expect that the topological two-forms do not contribute any propagating degrees of freedom to the theory, as the results from the canonical analysis of the two-form model in section 4.4 suggest. Similarly we do not yet have a good understanding of the meaning of the consistency condition (6.87) that follows from the $\mathcal{H}_{\mathrm{P}2}$ constraints.

The complete canonical constraint algebra of ExFT needs to be known in order to verify that the number of physical field space degrees of freedom is indeed 128, because we cannot otherwise know which of the canonical constraints are first class and which ones are second class functions.
It seems that based on the results of the canonical analysis of the five-dimensional ungauged maximal $\mathrm{E}_{6(6)}$ invariant supergravity of chapter 5 we should expect that the (bosonic) $\mathrm{E}_{6(6)}$ ExFT, without the two-forms, does indeed have 128 physical degrees of freedom. These 128 physical field space degrees of freedom consist of 5 that come from the external metric $G_{\mu\nu}$, while 42 come from the scalar fields M_{MN} and 81 are contributed by the generalised one-forms A_μ^M.[3] As is suggested by the results of the canonical analysis of the topological two-form model in section 4.4 the two-forms $B_{\mu\nu M}$ should not contribute any additional propagating physical degrees of freedom to ExFT. One might hence naively suspect that only the canonical constraints associated to the two-forms are second class functions, at least if the implicit treatment of the scalar coset constraints is used (cf. section 4.3) and in this case the counting of the physical degrees of freedom would work out, as described above, to be 128 — this however remains to be verified by the explicit computation of the complete canonical constraint algebra of the ExFT.

[2]The explicit appearance of the spin connection ω_{nab} in the relation (6.146) has been discussed in chapter 5 for the analogous five-dimensional relation.

[3]To arrive at the 42 degrees of freedom from the scalar fields the implicit coset constraints have to be taken into account, as was explained in section 4.3.

6.11 The generalised vielbein and USp(8)

In this chapter we have so far discussed the canonical analysis of the $E_{6(6)}$ exceptional field theory formulated in terms of the generalised metric M_{MN}. While the description in terms of the generalised metric is sufficient for the bosonic sector of ExFT it can be useful for some applications (e.g. coupling the theory to fermions or explicitly manifesting the USp(8) symmetry) to make use of the formulation of ExFT that is written in terms of the generalised USp(8) vielbein $\mathcal{V}_M^{AB}$. In this section we introduce the generalised USp(8) vielbein and discuss how it can be used to reformulate the canonical $E_{6(6)}$ ExFT.

In the fully supersymmetric $E_{6(6)}$ ExFT the generalised USp(8) vielbein is essential for coupling the theory to the fermions and in this section we use the same conventions for the USp(8) invariant form as the references [26, 36]. The unitary symplectic Lie group USp(8), which is the maximal compact subgroup of $E_{6(6)}$, is 36-dimensional and it has an 8-dimensional fundamental representation (cf. table 2.1). As indicated by the table 6.1 we use the capital indices $A, B, \ldots, F = 1, \ldots, 8$ to denote the fundamental representation of the USp(8).

For the $E_{6(6)}$ ExFT the internal generalised metric M_{MN} is an $E_{6(6)}$/USp(8) coset representative and hence there is a direct analogy between the internal metric M_{MN} and the external metric $G_{\mu\nu}$ (of general relativity), which is itself a GL(d)/SO($1, d-1$) coset representative, with d being the space-time dimension. In analogy to the local Lorentz invariance and the (external) vielbein (or frame field) $E_\mu{}^\alpha$ of the external metric (3.84) we can make use of the local USp(8) invariance to introduce a generalised internal vielbein $\mathcal{V}_M^{AB} = \mathcal{V}_M^{[AB]}$ defined by (6.148).

$$M_{MN} =: \mathcal{V}_M^{AB}\, \mathcal{V}_N^{CD}\, \Omega_{AC}\, \Omega_{BD} = \mathcal{V}_M^{AB}\, \mathcal{V}_{NAB} \tag{6.148}$$

Compared to the definition of the usual vielbein (3.84) the USp(8) symplectic form Ω_{AB} takes the place of the Minkowski metric in the definition (6.148). We define the lowered indices of $\mathcal{V}_{NAB}$ in (6.148) by $\mathcal{V}_{NAB} := \mathcal{V}_N^{CD}\Omega_{AC}\Omega_{BD}$. Furthermore we define the inverse symplectic form, as in section 3.4, by the condition $\Omega_{AB}\,\Omega^{CB} := \delta_A^C$, which is equivalent to $\Omega_{AB}\,\Omega^{BC} = -\delta_A^C$. Moreover the tracelessness condition (6.149) has to be satisfied by the generalised vielbein.

$$\mathcal{V}_M^{AB}\, \Omega_{AB} = 0 \tag{6.149}$$

The antisymmetric fundamental index pair $[AB]$ has 27 components, when the condition (6.149) is taken into account, which agrees with the dimension of the fundamental $E_{6(6)}$ representation. We can then define the inverse generalised vielbein by the conditions (6.150) and (6.151) as in reference [26].

$$\mathcal{V}_M^{AB}\, \mathcal{V}_{AB}^N := \delta_M^N \tag{6.150}$$

$$\mathcal{V}_M^{AB}\, \mathcal{V}_{CD}^M := \frac{1}{2}(\delta_C^A\delta_D^B - \delta_D^A\delta_C^B) - \frac{1}{8}\Omega^{AB}\,\Omega_{CD} \tag{6.151}$$

As in the generalised metric formulation we can compute the canonical momenta of the generalised vielbein, which only get contributions from the kinetic term of the scalar fields (6.11) and we find that they are given by (6.152).

$$\begin{aligned}\Pi^M_{AB}(\mathcal{V}) = \frac{e}{3N}\Big[&\dot{\mathcal{V}}^{CD}_N \mathcal{V}^M_{CD} \mathcal{V}^N_{AB} + \dot{\mathcal{V}}^{EF}_K \mathcal{V}^K_{CD} \mathcal{V}^N_{EF} \mathcal{V}^{MCD} \mathcal{V}_{NAB} \\ &+ \mathbb{L}_{A_t} \mathcal{V}^M_{AB} \quad + \mathbb{L}_{A_t} \mathcal{V}^N_{CD} \mathcal{V}_{NAB} \mathcal{V}^{MCD} \\ &+ N^n \mathcal{D}_n \mathcal{V}^M_{AB} + N^n \mathcal{D}_n \mathcal{V}^N_{CD} \mathcal{V}_{NAB} \mathcal{V}^{MCD}\Big]\end{aligned} \tag{6.152}$$

The canonical momenta of the generalised vielbein (6.152) can be related to the rescaled canonical momenta of the generalised metric by the identity (6.153).

$$\Pi^M_{AB}(\mathcal{V}) = 2\,\Pi^{MN}(M)\,\mathcal{V}_{NAB} \tag{6.153}$$

$$\Pi^{MN}(M) = \frac{1}{2}\Pi^{(M}_{AB}(\mathcal{V})\,\mathcal{V}^{N)AB} \tag{6.154}$$

We find that the inverse relation (6.154) is in form identical to the analogous identity (5.13) which relates the canonical momenta of the (external) metric to the canonical momenta of the (external) vielbein in general relativity.

Just like the canonical vielbein momenta imply the Lorentz constraints (6.39) in general relativity the above generalised vielbein momenta lead to the primary canonical USp(8)-constraints $\mathcal{H}_{\mathrm{USp(8)}}$ (6.155).

$$(\mathcal{H}_{\mathrm{USp(8)}})^{AD} := \mathcal{V}^{AB}_M\,\Omega_{BC}\,\Pi^{MCD} + \mathcal{V}^{DB}_M\,\Omega_{BC}\,\Pi^{MCA} = 0 \tag{6.155}$$

The contraction of the canonical momenta with the generalised vielbein in the USp(8)-constraints (6.155) is symmetrised, which is in contrast to the antisymmetric Lorentz constraints. Therefore the $(\mathcal{H}_{\mathrm{USp(8)}})^{AD}$ constraints consist of 36 independent components, which is equal to the dimension of the group USp(8).

We can define the fundamental Poisson bracket of the generalised vielbein with its canonical momenta by (6.156).

$$\{\mathcal{V}^{AB}_M, \Pi^M_{CD}(\mathcal{V})\} := \frac{1}{2}(\delta^A_C\delta^B_D - \delta^A_D\delta^B_C) - \frac{1}{8}\Omega^{AB}\,\Omega_{CD} \tag{6.156}$$

We should furthermore verify that the Legendre transformation is indeed invariant under this change of canonical coordinates before we are able to rewrite the ExFT Hamiltonian (6.64) in terms of the generalised vielbein $\mathcal{V}^{AB}_M$. We can confirm that the Legendre transformation is indeed invariant (6.157) and the additional symplectic term that originates from the identity (6.151) vanishes because of (6.149).

$$\frac{1}{2}\sum_{R,S=1,\dots,27} \dot{M}_{RS}\cdot\tilde{\Pi}^{RS}(M) = \frac{1}{2}\sum_{\substack{M=1,\dots,27\\ A,B=1,\dots,8}} \dot{\mathcal{V}}^{AB}_M\cdot\Pi^M_{AB}(\mathcal{V}) \tag{6.157}$$

Because of the equivalence (6.157) we can now replace the generalised metric and its canonical momenta in the Hamiltonian (6.64) by making use of the identities for the generalised vielbein (6.148) and their canonical momenta (6.154) to replace the canonical variables. The contributions to the Hamiltonian from the scalar kinetic term (6.44) can then be rewritten in terms of the generalised vielbein as in (6.158).

$$
\begin{aligned}
&\frac{1}{2}\sum_{\substack{M=1,\ldots,27\\A,B=1,\ldots,8}} \dot{\mathcal{V}}_M^{AB}\cdot\Pi^M_{AB}(\mathcal{V})-\mathcal{L}_{\rm sc} \qquad (6.158)\\
&= N\cdot\Big[\frac{3}{16e}\Pi^M_{AB}\Pi^S_{CD}\mathcal{V}^{CD}_M\mathcal{V}^{AB}_S+\frac{3}{16e}\Pi^M_{AB}\Pi^R_{CD}\mathcal{V}^{SCD}\mathcal{V}^{AB}_S\mathcal{V}^{EF}_M\mathcal{V}_{REF}\\
&\qquad -\frac{e}{12}g^{kl}\mathcal{D}_k\mathcal{V}^{AB}_M\mathcal{D}_l\mathcal{V}^M_{AB}-\frac{e}{12}g^{kl}\mathcal{D}_k\mathcal{V}^{AB}_M\mathcal{D}_l\mathcal{V}^N_{CD}\mathcal{V}_{NAB}\mathcal{V}^{MCD}\Big]\\
&+N^l\cdot\Big[\frac{1}{4}\Pi^M_{AB}\mathcal{D}_l\mathcal{V}^{AB}_M+\frac{1}{4}\Pi^M_{AB}\mathcal{D}_l\mathcal{V}_{NCD}\mathcal{V}^{NAB}\mathcal{V}^{CD}_M\Big]\\
&+A^K_t\cdot\Big[\frac{1}{4}\Pi^M_{AB}(\partial_K\mathcal{V}^{AB}_M-6\,\mathbb{P}^{P\ L}_{M\ K}\partial_L\mathcal{V}^{AB}_P)\\
&\qquad +\frac{1}{4}\Pi^M_{AB}(\partial_K\mathcal{V}^{CD}_N-6\,\mathbb{P}^{P\ L}_{N\ K}\partial_L\mathcal{V}^{CD}_P)\mathcal{V}^{NAB}\mathcal{V}_{MCD}\Big]
\end{aligned}
$$

It seems to be a general feature of the generalised USp(8) vielbein formulation that there are always two differently contracted versions of each term of (6.44) in the rewriting (6.158). In these terms of (6.158) we can moreover note the appearance of the inverse vielbein, which seems to necessary and which makes this expression non-polynomial.

Chapter 7

Summary, conclusions and outlook

In this thesis we have constructed and investigated the canonical formulations of the bosonic $E_{6(6)}$ exceptional field theory and of the bosonic sector of the unique manifestly $E_{6(6)}$ invariant ungauged maximal five-dimensional supergravity theory. We have calculated the explicit non-integral form of the topological term of the $E_{6(6)}$ ExFT, which we needed in order to carry out the Legendre transformation of the theory, and examined a topological model theory based on the two-form kinetic term, which lead us to identify problems related to the construction of a Dirac bracket in the extended generalised geometry. To illustrate the construction of an extended generalised geometry we explicitly constructed the Y-tensor for the symplectic group $\mathrm{Sp}(2n)$. Furthermore we established a simplified canonical treatment of the scalar coset constraints, which we explained by taking the $\mathrm{SL}(n)/\mathrm{SO}(n)$ coset as an example. The canonical formulation of the manifestly $E_{6(6)}$ invariant ungauged maximal five-dimensional supergravity theory was constructed and we carried out a comprehensive canonical analysis, which included all gauge transformations and the full constraint algebra. This canonical analysis in particular provided crucial insights into the treatment of the topological term, which we applied in the construction of the canonical formulation of the $E_{6(6)}$ ExFT. The full canonical Hamiltonian of the $E_{6(6)}$ ExFT was constructed and most of the canonical (gauge) transformations and parts of the constraint algebra were calculated. Moreover we examined how the canonical $E_{6(6)}$ ExFT can be formulated in terms of the generalised USp(8) vielbein.
In this chapter we summarise and discuss the main findings of this thesis, as well as the remaining open questions. Furthermore we give an outlook on some of the possible directions of future research.[1]

Background

In the chapters 2, 3 and 4 we reviewed the background knowledge concerning (gauged maximal) supergravity, extended generalised (exceptional) geometry, exceptional field theory and the canonical analysis of constrained Hamiltonian systems. In chapter 2 we discussed the hidden exceptional global symmetries that emerge in the toroidal compactifications of eleven-dimensional supergravity, how group invariant supergravity Lagrangians can be constructed and how the tensor hierarchy in gauged maximal supergravity arises due to the non-trivial Jacobiator of the structure constants of the gauge group. This information became useful later on to understand the structure of the manifestly $E_{6(6)}$ invariant formulation of five-dimensional supergravity and to see the analogy between the structure of the tensor hierarchy of ExFT and gauged supergravity. The concept of geometrisation was illustrated in chapter 3 by looking at Kaluza-Klein theory, in which four-dimensional Maxwell theory and a massless scalar

[1] This chapter follows the structure of parts of the publications [40, 41].

field become geometrised and can be seen as arising from five-dimensional gravity. Next we reviewed how extended generalised geometries can be constructed for various symmetry groups, e.g. $\mathrm{O}(n,n)$, $\mathrm{Sp}(2n)$ or $\mathrm{E}_{n(n)}$ and we explicitly constructed the Y-tensor of the extended generalised $\mathrm{Sp}(2n)$ symplectic geometry as an example. We discussed how the extended generalised geometries can be used to manifest larger symmetry groups and thereby geometrise part of the original degrees of freedom. Moreover we reviewed how the section condition of ExFT can be solved in order to reduce the theory to, among others, eleven-dimensional supergravity or the manifestly $\mathrm{E}_{6(6)}$ invariant maximal five-dimensional supergravity. In chapter 4 we reviewed the fundamental theory of constrained Hamiltonian systems, which formed the methodological basis for the following chapters.

The topological term of the $\mathrm{E}_{6(6)}$ ExFT

In the Lagrangian formulation of the $\mathrm{E}_{6(6)}$ ExFT [3, 25, 26, 36] the topological term is only stated in the manifestly covariant form, where it is written as a $(6+27)$-dimensional integral over an exact six-form, although the variation of the explicit term is also given. But in order to calculate the Legendre transformation of the $\mathrm{E}_{6(6)}$ ExFT Lagrangian, in which the topological term is mixed with the other transformation terms, the explicit non-integral $(5+27)$-dimensional (non-manifestly gauge invariant) form of the topological term is needed. Noting the similarity of gauged maximal supergravity in five dimensions to the $\mathrm{E}_{6(6)}$ ExFT we constructed an ansatz for the ExFT topological term based on the topological term of the five-dimensional gauged supergravity [84]. The coefficients in this ansatz were then fixed by computing its general variation, which we compared to the variation stated in [25]. The explicit form of the topological term (3.135) which was thus found is very complicated. Because this complexity results in a very large number of contributions to many of the calculations in the canonical analysis of the theory, the topological term presents one of the main computational challenges in the canonical formulation of the $\mathrm{E}_{6(6)}$ ExFT. There does not seem to be any possibility of avoiding this complication in the $\mathrm{E}_{6(6)}$ ExFT. Due to an exceedingly large number of topological contributions some of the transformations of the modified momenta $\mathcal{P}^m_M$ have not been fully computed and consequently many of the constraint Poisson algebra relations have also not been (fully) computed. With the assistance of a suitable computer algebra program, that is able to simultaneously handle the multitude of mathematical structures involved in the canonical formulation of ExFT, it should be possible to perform these calculations in full, because the complexity of the topological contributions is a purely computational issue. For the canonical quantisation of ExFT, where a simpler Hamiltonian topological term is certainly desirable, one might alternatively want to consider the $\mathrm{E}_{8(8)}$ ExFT [149], whose topological term is already explicitly known in a non-integral form and which is perhaps (slightly) simpler than that of the $\mathrm{E}_{6(6)}$ ExFT, although this would introduce new complications e.g. in the form of the more complicated group $\mathrm{E}_{8(8)}$ and constrained compensator fields.

Topological two-forms model theory and the Dirac bracket in ExFT

The existence of the two-form fields $B_{\mu\nu M}$ in the tensor hierarchy of the $\mathrm{E}_{6(6)}$ ExFT is necessary in order to absorb the non-covariance in the transformation (3.88) of the one-form field strength and this is in direct analogy to the tensor hierarchy of gauged supergravity. Because there are no degrees of freedom, coming from eleven-dimensional supergravity, left to make these two-form we are required to introduce them as topological fields and their dynamics is governed entirely by the topological term (3.135). To better understand the properties of these two-forms, we investigated the canonical formulation of a model theory consisting of just the isolated two-form kinetic term, which is part of the topological term of the $E_{6(6)}$ ExFT, in section 4.4. In the canonical analysis we confirmed that there are no propagating degrees of freedom in this model and we moreover found that because the kinetic term of the two-forms is topological the two-forms do not canonically transform under external diffeomorphisms. This fact remains true in the full ExFT because the two-forms only couple to the external metric through the (non-kinetic) Stückelberg coupling terms in the one-form field strength.

In the topological model theory we moreover saw that the two-form tensor gauge transformations appear in a very different form canonically. We can compare the canonical two-form tensor gauge transformations to the canonical one-form gauge transformations of Maxwell theory, which initially also appear as shift transformations, but which can be translated into the usual Lagrangian form $\delta_\lambda A_\mu = \partial_\mu \lambda$ by means of the extended Hamiltonian formalism (this procedure has been described in chapter 19 of reference [28]). What we have found for the topological two-forms however, is that all canonical constraints of the model are second class functions and hence the extended Hamiltonian is identical to the total Hamiltonian and therefore an approach analogous to the procedure of reference [28] does not work, because there are no arbitrary parameters in the Hamiltonian. Three-dimensional Chern-Simons theory is very similar to this two-form model in the sense that it is also a topological theory with a Lagrangian that is linear in the field strength (and hence in the time derivative). This similarity leads to a very similar canonical constraint structure and through a better understanding of this analogy one might hope to identify a procedure that would make it possible to cast the tensor gauge transformations of the two-form model into the standard form.

In section 4.4 we proposed the definition (4.86) for a Dirac bracket in an extended generalised exceptional geometry, which is needed in order to manage the second class constraints of the two-form model. The non-constraint terms in the Poisson algebra of the two-form model are not constant and they depend on both external and internal derivatives. To calculate the explicit expression for the Dirac bracket we consequently found that we would need to solve equations of the form (4.88) for a primitive of the $(5 + 27)$-dimensional Dirac delta distribution. It is not clear how such a primitive distribution can be identified, but if it was known one should be able to compute the explicit form of the Dirac bracket for this model and for the full ExFT, which would likely also clarify the structure of the tensor gauge transformations and the meaning of the two-form constraints in the ExFT.

Canonical treatment of the scalar coset constraints

In section 4.3 we discussed how the second class scalar $\mathrm{SL}(n)/\mathrm{SO}(n)$ coset constraints can be treated explicitly by adding the coset constraints with Lagrange multipliers to the Lagrangian and constructed the Dirac bracket of this model. We then explained how one can treat the coset constraints implicitly, thereby avoiding the need for a Dirac bracket. In chapter 5 and chapter 6 the much more complicated scalar $E_{6(6)}/\mathrm{USp}(8)$ coset constraints were treated in the implicit formalism. The implicit formalism of the coset constraints significantly simplified the canonical analysis of the manifestly $E_{6(6)}$ invariant five-dimensional supergravity, because it meant that all canonical constraints are first class functions and there was no need to construct a Dirac bracket for this theory. Nonetheless we recovered the correct dynamics for all fields of this theory in the implicit formalism. In the canonical analysis of the $E_{6(6)}$ ExFT the benefits of this treatment are less substantial due to the second class two-form constraints, which necessarily require the introduction of a Dirac bracket, as was discussed above.

Canonical $\mathbf{E}_{6(6)}$ invariant five-dimensional supergravity

In chapter 5 we carried out a comprehensive canonical analysis of the bosonic sector of the unique manifestly $E_{6(6)}$ invariant ungauged maximal five-dimensional supergravity theory. We constructed all canonical momenta, the extended canonical Hamiltonian, all canonical gauge transformation, the complete Poisson algebra of the canonical constraints and re-derived the number of physical degrees of freedom. The manifestly $E_{6(6)}$ invariant five-dimensional supergravity is related to the $E_{6(6)}$ ExFT via the trivial solution of the section condition, which eliminates all internal coordinate dependence in the ExFT. Because of this close relation of the theory to the ExFT we were able to derive some crucial insights from this canonical analysis and perhaps the most important lesson was the canonical treatment of the topological term. We were able to simplify the Hamiltonian and the form of the gauge transformations by introducing modified one-form momenta, where the topological contribution to the canonical momenta has been subtracted. While this (non-canonical) redefinition resulted in a simpler Hamiltonian it complicated the analysis because the modified momenta do not Poisson-commute amongst each other, but we were able to carry out all calculations by working in orders of the topological term coefficient. We were able to apply all of these insights in the canonical analysis of the $E_{6(6)}$ ExFT and knowing the right canonical coordinates greatly simplified the Legendre transformation. Because the canonical analysis of the $E_{6(6)}$ invariant five-dimensional supergravity has been carried out in a general and comprehensive form these results may also be useful to applications that are unrelated to ExFT.

$\mathbf{E}_{6(6)}$ ExFT Hamiltonian and canonical constraints

In chapter 6 we calculated the ADM decomposition of the $E_{6(6)}$ ExFT Lagrangian, all canonical momenta and the Legendre transformation to find the canonical Hamiltonian of $E_{6(6)}$ ExFT (6.64) (see also equation (7.1)), which is one of the main results of this thesis. The most concise canonical formulation of ExFT was found to be given in terms of the modified one-form momenta-like variables $\mathcal{P}^m_M(A)$ (6.21) where, in analogy to chapter 5, all of the topological contributions to the canonical momenta $\Pi^m_M(A)$ have been subtracted. The modified momenta do not Poisson-commute $\{\mathcal{P}^m_M(A), \mathcal{P}^n_N(A)\} \neq 0$ because the redefinition (6.21) is not a canonical transformation, nonetheless they seem to be the most natural variables to use to write the canonical (gauge) transformations and the Hamiltonian. This Poisson non-commutativity is a further computational complication in the canonical analysis of the theory, which originates from the complexity of the topological term (3.135).
The canonical Hamiltonian (6.64) of the $E_{6(6)}$ ExFT is found to be given by the covariantisation of the canonical Hamiltonian (5.19) of the manifestly $E_{6(6)}$ invariant five-dimensional supergravity, but there are further purely internal terms, such as e.g. the Hamiltonian scalar potential. When written on the primary constraint surface, the canonical Hamiltonian (6.64) of the $E_{6(6)}$ ExFT can be written in the form (7.1) and consists of the Hamilton constraint $\mathcal{H}_{\text{Ham}}$, the (external) diffeomorphism constraints $\mathcal{H}_{\text{Diff}}$, the generalised diffeomorphism constraints $\mathcal{H}_{\text{GD}}$ and the two-form tensor gauge

constraints $\mathcal{H}_{\mathrm{S1}}$.

$$
\begin{aligned}
\mathcal{H}_{\mathrm{ExFT}} = &+ N \cdot \Big[+ \frac{1}{4e}\Pi_{ab}(e)\,\Pi_{ab}(e) - \frac{1}{12e}\Pi(e)^2 - e\,\hat{R} + e\,V_{\mathrm{HP}} \\
&\qquad + \frac{3}{2e}\Pi^{MN}(M)\,\Pi_{MN}(M) - \frac{e}{24}\,\mathcal{D}_m M_{MN}\,\mathcal{D}^m M^{MN} \\
&\qquad + \frac{e}{4}\,\mathcal{F}^{mn}_{M}\,\mathcal{F}^{M}_{mn} + \frac{1}{2e}\,\mathcal{P}^{m}_{M}\,\mathcal{P}^{M}_{m} \Big] \\
&+ N^n \cdot \Big[+ 2\,\Pi^{m}{}_{a}(e)\,\mathcal{D}_{[n} e_{m]a} - e_{na}\,\mathcal{D}_m \Pi^{m}{}_{a}(e) \\
&\qquad + \frac{1}{2}\Pi^{MN}(M)\,\mathcal{D}_n M_{MN} \\
&\qquad + \mathcal{F}^{M}_{nl}\mathcal{P}^{l}_{M} + \partial_M\left(g_{mn}\,M^{MN}\,\mathcal{P}^{m}_{N}\right) \Big] \\
&+ A^M_t \cdot \Big[- \mathcal{D}_l\mathcal{P}^{l}_{M} - 5\,d^{NLS}\,d_{MNK}\,A^K_m\partial_S\mathcal{P}^{m}_{L} + (\mathcal{H}_{top})_M \\
&\qquad + \Pi^{m}{}_{a}(e)\,\partial_M e_{ma} - \frac{1}{3}\partial_M\Pi(e) \\
&\qquad + \frac{1}{2}\,\Pi^{KL}(M)\,\partial_M M_{KL} - 6\,\mathbb{P}^{R}{}_{K}{}^{S}{}_{M}\,\partial_S\left(\Pi^{KL}(M)\,M_{RL}\right) \Big] \\
&+ B_{tlM} \cdot \Big[+ 10\,d^{MKL}\partial_K\left(\mathcal{P}^{l}_{L} - \kappa\,\epsilon^{lmnr}\,\mathcal{H}_{mnrL}\right) \Big]
\end{aligned}
\tag{7.1}
$$

The scalar potential (6.66) remains largely unchanged in the Legendre transformation and only a single potential term cancels. In comparison to five-dimensional supergravity the Hamiltonian scalar potential is the main addition to the Hamilton constraint, besides the covariantisation terms. Because the one-form fields A^M_μ of ExFT are used to gauge the generalised diffeomorphism symmetry, the generalised diffeomorphism constraints, which are associated to the Lagrange multiplier A^M_t, can be seen as an extension of the abelian $\mathrm{U}(1)^{27}$ or Gauß constraints of the five-dimensional supergravity. Inside the generalised diffeomorphism constraints we can find the extensive Hamiltonian topological term (6.68), which in particular contains the two-form kinetic term. In the Lagrangian formulation the duality between the differential forms is implied by the on-shell duality relation (3.113), which is the Lagrangian equation of motion of the two-forms. The duality relation (3.113) is in form very similar to the secondary second class two-form tensor gauge constraints $\mathcal{H}_{\mathrm{S1}}$ and we may suspect that the duality of the one- and two-forms is canonically implied by the transformations that the two-form constraints generate. Naively one would expect the time evolution of the two-form canonical momenta to be equivalent to the Lagrangian equation of motion of the two-forms, but this Poisson bracket does not seem to be of the right form. The precise details of this duality are not yet fully understood canonically and may require the Dirac bracket.

We confirmed that the above secondary constraints are indeed required by verifying the consistency of the primary constraints of shift type, i.e. those that take the form of vanishing canonical momenta. Because the Lagrangian two-form kinetic term contains only a single time derivative the primary two-form constraints $\mathcal{H}_{\mathrm{P2}}$ are not of shift type and instead directly relate the spatial components of the two-form to their canonical momenta. The consistency of the primary $\mathcal{H}_{\mathrm{P2}}$ constraints requires the condition (6.87), which implies the existence of secondary constraints that depend on the Lagrange multipliers. These constraints are in analogy to the $\mathcal{H}_{\mathrm{S2}}$ constraints of the topological two-form model theory from section 4.4, which depend on the Lagrange multipliers B_{tnM}. Because the Lagrange multipliers N, N^n, A_t^M and B_{tnM} of ExFT are independent one should be able to divide the consistency condition (6.87) into several independent constraints, which take the form of the transformations generated on the canonical momenta $\Pi^{klM}(B)$ but with the transformation parameters given by the Lagrange multipliers. The explicit form of the transformations that these constraints generate canonically is not very illuminating, at least without the use of the proper Dirac bracket, but in analogy to the model theory from section 4.4 we should probably expect that these transformations are ultimately related to the tensor gauge transformations of the two-forms $B_{\mu\nu M}$. The origin of these constraints can be traced back to the fact that the Lagrangian two-form kinetic term is linear in the time derivative combined with the fact that the two-forms only couple to the secondary constraints via the Stückelberg coupling and the topological term couplings. A better understanding of the topological two-form model from section 4.4, including the explicit construction of the Dirac bracket, is needed to properly understand the role of the two-form constraints in the full $E_{6(6)}$ ExFT.

Canonical (gauge) transformations of the $E_{6(6)}$ ExFT

Most of the transformations that are generated by the canonical constraints of the $E_{6(6)}$ ExFT, via the Poisson brackets, have been calculated in section 6.9. Due to the complexity of the topological term of the $E_{6(6)}$ ExFT the full transformations of the modified momenta $\mathcal{P}_M^m$ are very difficult to compute and hence some of these transformations have only been calculated for the $\kappa = 0$ case, in which the topological contributions are omitted. In this thesis the $\kappa = 0$ case is only considered as a computational tool and one should not expect this limit to correspond to any meaningful physical theory upon solution of the section condition. Nonetheless we have found that the form of most results remains largely unchanged in the $\kappa = 0$ limit, with the notable exception of the purely topological two-form dynamics.

We found that the generalised diffeomorphism constraints, which are associated to the Lagrange multiplier A_μ^M, do indeed generate generalised diffeomorphism transformations and all fields and canonical momenta transform as the generalised Lie derivative — although this is only true on the primary constraint surface for the two-forms. Furthermore we found that additional tensor gauge transformation terms appear in the transformations of the one-forms and their conjugate momenta. These tensor gauge transformations and $\mathcal{H}_{\mathrm{S1}}$ constraint terms in the transformations generated by the $\mathcal{H}_{\mathrm{GD}}$ constraints can be seen as being analogous to the one-form gauge transformations that appear in the transformations generated by the $\mathcal{H}_{\mathrm{Diff}}$ constraints in $E_{6(6)}$ invariant five-dimensional supergravity. Similarly they can be seen as being analogous to the $\mathcal{H}_{\mathrm{GD}}$ constraint terms which appear in the transformation generated by the $\mathcal{H}_{\mathrm{Diff}}$ constraints in ExFT, because the $\mathcal{H}_{\mathrm{GD}}$ constraints are analogous to the $\mathcal{H}_{\mathrm{Gauß}}$ constraints of the five-dimensional supergravity which are associated to the same Lagrange multiplier. The transformations of the two-forms under the $\mathcal{H}_{\mathrm{GD}}$ constraints can only be written as the generalised Lie derivative if the primary $\mathcal{H}_{\mathrm{P2}}$ constraints are applied and we found that there are additional contributions, which may be related to the tensor gauge transformations. Because the Lagrangian kinetic term of the two-forms is linear in the time derivative the two-forms do generally not transform under the secondary constraints of the Hamiltonian.

We found that the external diffeomorphism constraints $\mathcal{H}_{\mathrm{Diff}}$ generate transformations on the external vielbein and the scalar fields which are of the form of the covariantised external Lie derivative. This is in agreement with the covariantised external diffeomorphisms of the Lagrangian formulation of the $E_{6(6)}$ ExFT. The form of the external diffeomorphism transformation of the one-forms (6.110) agrees with the Lagrangian gauge transformation (3.112), but surprisingly the sign of the $\partial_M \xi^n$ term, which is responsible for determining the coefficients in the Lagrangian, differs. In section 6.9.3 we showed that this sign originates canonically from the ADM decomposition of the Einstein-Hilbert improvement term. Furthermore we showed in section 3.5 that the analogous sign in the Lagrangian formulation originates from a compensating Lorentz transformation and this calculation has already been carried out in [25, 103] with the same result. In both formalisms the calculation of the sign of this $\partial_M \xi^n$ term does not seem to involve any choices. In particular there does not seem to be any choice with respect to the Lorentz transformations and the conventions of both formulations, e.g. for the Minkowski metric, also seem to agree. We thus do not have an explanation for this difference of the sign of the $\partial_M \xi^n$ term. The $\mathcal{H}_{\text{Diff}}$ constraints generate transformations on the canonical momenta conjugate to the vielbein, the scalar fields and the one-forms, which are of the form of the covariantised external Lie derivative, but there are additional $\partial_M \xi^n$ terms. In analogy to the Lagrangian formulation it should be expected that they take a similar role in determining the relative coefficients in the Lagrangian or equivalently in determining the coefficients in the Hamiltonian by mixing the field dependence of terms — one might thus presume that these $\partial_M \xi^n$ terms lead to precise cancellations in the Poisson algebra of the constraints. The $\mathcal{H}_{\text{Diff}}$ constraints furthermore induce additional $\mathcal{H}_{\text{GD}}$ and $\mathcal{H}_{\text{S1}}$ constraint terms in the transformation of the one-form momenta.

We found that the canonical time evolution of the vielbein, scalar fields and one-forms, as generated by the Hamilton constraint $\mathcal{H}_{\text{Ham}}$, takes the same form as the analogous time evolutions in five-dimensional $E_{6(6)}$ invariant supergravity. Because the two-forms do not transform canonically under the secondary constraints of the Hamiltonian their time evolution has to be given entirely in terms of gauge transformations. Due to the Hamiltonian scalar potential, the covariant derivatives and the modified one-form momenta term in the Hamilton constraint, the time evolution of the canonical momenta in ExFT is significantly more complicated than the time evolution of the canonical momenta in the five-dimensional $E_{6(6)}$ invariant supergravity. When the trivial solution of the section condition is applied the expressions we found for the time evolutions reduce to those of the five-dimensional theory. However the expressions for the canonical equations of motion stated in this thesis are likely not written in the simplest and most covariant form and it should be constructive to compare these expressions to the decomposition of the analogous Euler-Lagrange equations to find a simpler form.

The transformations generated by the $\mathcal{H}_{\text{S1}}$ constraints are identical to those of the topological model theory, however we were able to identify the transformation that they generate on the one-forms with the Lagrangian tensor gauge transformation that is induced in the one-forms (3.106) by the Stückelberg coupling.

Poisson algebra of the $E_{6(6)}$ ExFT canonical constraints

In section 6.10 we investigated the Poisson algebra of the canonical constraints of the $E_{6(6)}$ ExFT. The Poisson brackets of the Lorentz constraints with the secondary constraints were computed and we found that the section condition was not needed in these calculations. The Poisson brackets of the second class two-form constraints were computed in section 4.4 for the topological model theory and require the introduction of a Dirac bracket, as discussed above. The Poisson bracket of the generalised diffeomorphism constraints with themselves was computed at $\kappa = 0$ and we were able to verify that the correct generalised diffeomorphism constraint term appears, but further canonical constraint terms may appear in this relation canonically. The canonical Poisson algebra does not necessarily need to take the same form as the Lagrangian gauge algebra, because it is possible to change the basis of the algebra. The Poisson algebra should however reduce to the Poisson algebra of the manifestly $E_{6(6)}$ invariant five-dimensional supergravity, which we fully computed in chapter 5. Combining this information with the Lagrangian gauge algebra relations, which have been described in [25, 36] and which we reviewed in section 3.5.5, we were able to give speculative results for some of the remaining Poisson brackets. The calculation of the full $E_{6(6)}$ ExFT Poisson algebra is computationally very difficult, in particular due to the topological terms, but also due to the complexity of the secondary constraints.

Role of the section condition in canonical $E_{6(6)}$ ExFT

The section condition (3.72) is not needed in the construction of the canonical Hamiltonian of the $E_{6(6)}$ ExFT. In the calculation of the canonical (gauge) transformations we found that the section condition was only needed once in order to match the two-form terms of the $\mathcal{H}_{\text{GD}}$ constraints to the equivalent terms that appear in the transformation (6.112) of the modified momenta $\mathcal{P}^m_M$ under the external diffeomorphism constraints. It may be possible to avoid the use of the section condition in the calculation of the transformations if there exists some ambiguity in the definition of the topological term, which could allow the addition of a term that vanishes under the section condition without affecting the overall variation of the topological term. In the formulation presented in this thesis the section condition has to be used in the calculation of (6.112) and such a modification of the topological term remains to be investigated. As far as the calculation of the Poisson algebra of the canonical constraints of ExFT has been carried out in this thesis, the section condition was only needed in the computation of the Poisson bracket of the generalised diffeomorphisms constraints with themselves (6.143), where it has to be applied many times and this was expected from the analogous Lagrangian relation. In this canonical analysis we have not found any natural physical interpretation for the section condition and the section condition has to be postulated ad hoc in the canonical formalism. The section condition cannot be interpreted as a canonical constraint because it restricts the (internal) coordinate dependence and not the canonical variables. For the same reason the section condition cannot be explicitly added to the Lagrangian.

Outlook

In order to fully complete the canonical analysis of the $E_{6(6)}$ ExFT the following questions should be addressed. The full transformations of $\mathcal{P}^m_M$, including all topological contributions, remain to be computed and the role of the two-form H_{S2} constraints remain to be fully understood, this also requires the explicit construction of the Dirac bracket of the ExFT. Moreover one should calculate the missing relations of the canonical Poisson algebra, including all topological contributions. Finally the explanation for the difference in the sign of the transformation (6.110) with respect to the Lagrangian formulation remains to be found.

Once all of these questions have been answered one should be able to proceed with the canonical quantisation procedure as described in [27, 28]. As was mentioned in chapter 1, some $E_{n(n)}$ invariant ExFT amplitudes have already been computed up to three loops in [33–35] for geometries that are of the particular form of Minkowski space times a torus. Furthermore the geometric quantisation of double field theory

has very recently been investigated in [32] and the extension of these results to ExFT has been briefly commented on.
Another possibly interesting direction of research would be to investigate the cosmological implications of ExFT in the canonical formulation and to study the local initial value problem of the extended generalised exceptional geometry, which has to obey the section condition. In analogy to the results of the canonical analysis of double field theory [31] one could try to conceive a generalised notion of asymptotic flatness and generalised definitions of ADM charges for a non-compact extended generalised exceptional geometry.
Finally it may be interesting to try to generalise the definition of the canonical Ashtekar phase space variables of general relativity. The Ashtekar variables (or Ashtekar connection) A_{ma} of general relativity were first discovered by Abhay Ashtekar in 1986 [37, 38]. They are an alternative set of phase space coordinates for four-dimensional (or three-dimensional) general relativity and when written in terms of the Ashtekar connection the canonical constraints of general relativity are of polynomial form, which means in particular that their inverses do not appear. The Ashtekar connection is moreover one of the basic ingredients of the loop quantum gravity (LQG) [213] approach to the quantisation of general relativity and a basic introduction to both topics can also be found in [214]. The Ashtekar connection of general relativity can take the form of (7.2), where ω_{mab} is the spatial spin connection, $\Pi^m{}_a(e)$ are the canonical momenta of the spatial dreibein $e_m{}^a$, γ is a constant (Barbero-Immirzi parameter) and ϵ_{abc} is the spatial Levi-Civita symbol [214].

$$A_{ma} := -\frac{1}{2}\,\epsilon_{abc}\,\omega_{mbc} + \frac{\gamma}{e}(\Pi_{ma}(e) - \frac{1}{2}e_{ma}\,\Pi(e)) \tag{7.2}$$

The inverse densitised dreibein $\tilde{E}_a{}^m := e\,e_a{}^m$ are the canonical variables conjugate to the Ashtekar connection [214]. When analysing eleven-dimensional supergravity, written in an external-internal split in the SO(16) =K($E_{8(8)}$) invariant form [22], it was discovered in reference [39] that the internal vielbein (which combines both metric and three-form degrees of freedom) behaves in a way that is similar to the $\tilde{E}_a{}^m$ variables. It was found that these variables in particular lead to supersymmetry constraints and transformations that are of polynomial form [39]. It was thus argued in [39, 73, 215] that generalised Ashtekar variables, which are canonically conjugate to the internal vielbein, might exist in the context of duality-covariant reformulations of eleven-dimensional supergravity. The internal vielbein of [39] should be seen as part of the generalised internal vielbein of the $E_{8(8)}$ ExFT and therefore one might hope to find a similar structure in the canonical formulation of ExFT.
The rewriting of the canonical formulation of the $E_{6(6)}$ ExFT in terms of the internal generalised USp(8) vielbein $\mathcal{V}_M^{AB}$ has been investigated in section 6.11. In section 6.11 we found that the (naive) generalised USp(8) vielbein $\mathcal{V}_M^{AB}$ does not have the desired properties of the Ashtekar connection and in particular we found that these variables do not lead to fully polynomial canonical constraints — although the generalised vielbein momenta terms in the Hamilton constraint (6.158) and the scalar potential, which we might regard as an "internal Ricci scalar", look somewhat like the Hamilton constraint of the ADM formulation of general relativity. It may nonetheless be interesting to attempt to identify a possible definition for a generalised Ashtekar connection and perhaps this may require a modification of the generalised vielbein. As the SO(16) =K($E_{8(8)}$) covariant results of [39] seem to suggest, the construction of a generalised Ashtekar connection could depend on the dimension of the geometry and hence it may be beneficial to examine the canonical formulation of the $E_{8(8)}$ ExFT.

If a generalised Ashtekar connection for ExFT could be found, one might be able to pursue a non-perturbative and background independent quantisation approach to ExFT similar to that of loop quantum gravity [213, 214]. A similar approach was taken in the references [216–218], where the background independent quantisation of eleven-dimensional supergravity was investigated using methods borrowed from loop quantum gravity — though without the use of a generalised Ashtekar connection and not in a duality-covariant formulation. One might moreover find that the quantised ExFT is not equivalent to supergravity anymore and this is perhaps more likely if the theory is quantised in terms of unusual canonical coordinates, such as a generalised Ashtekar connection.

Because the existence of a generalised Ashtekar connection remains speculative the canonical quantisation of ExFT, in the present set of canonical coordinates, may seem more attainable.

Appendix A

Useful formulæ

In this appendix we explain some useful formulæ that are needed in the calculations of the chapters 4, 5 and 6. This appendix is based on parts of the publication [40].

A.1 Schouten identity

We can exploit the fact that we can "over-antisymmetrise" a tensor to make it vanish, e.g. we can take a set of $(d+1)$ indices in d dimensions and antisymmetrise them to get zero (here given in four dimensions). Where ϵ^{lmnr} is the Levi-Civita symbol in four dimension and v^k are vector components.

$$\epsilon^{[lmnr}\ v^{k]} = 0 \tag{A.1}$$

If we expand this expression we get the following useful identity.

$$\epsilon^{lmnr}\ v^k = -4\,\epsilon^{k[lmn}\ v^{r]} \tag{A.2}$$

In other words the statement of the identity is that there cannot be $(d+1)$ independent vectors in d dimensions.

A.2 Delta distribution identities

Here we list some useful identities concerning the Dirac delta distribution.

$$\delta(x-y) = \delta(y-x) \tag{A.3}$$

$$\delta'(x-y) = -\,\delta'(y-x) \tag{A.4}$$

$$\partial_{(x)}\delta(x-y) = \left.\frac{d\delta(z)}{dz}\right|_{z=x-y}\frac{d(x-y)}{dx} = -\,\partial_{(y)}\delta(x-y) \tag{A.5}$$

$$\partial_{(x)}\delta(x-y) = \left.\frac{d\delta(\tilde{z})}{d\tilde{z}}\right|_{\tilde{z}=-z=y-x}\frac{d(y-x)}{dx} = \partial_{(x)}\delta(y-x) \tag{A.6}$$

A.3 Vielbein determinant derivatives

From the definition of the determinant and the fact that its variation is given by $\delta e = e\, e_n^a\, \delta e_a^n$ we find the following identity.

$$e_a^n\,\partial_p e_n^a = e^{-1}\,\partial_p e = -e\,\partial_p e^{-1} \tag{A.7}$$

Appendix B

Additional Poisson bracket relations for chapter 5

We list some Poisson bracket relations in this appendix that are either intermediate results or formulae that can be derived from more fundamental statements in chapter 5, but which may not be immediately obvious. This appendix is based on parts of the publication [40].

$$\{M^{MN}, \Pi^{PQ}(M)\} = \left(-M^{MP}\, M^{QN} - M^{MQ}\, M^{PN}\right) \delta^{(4)}(x-y) \tag{B.1}$$

$$\begin{aligned}\int d^4x\, \phi(x)\, \{-e\, R(x), \Pi^n{}_a(e)(y)\} =& + 2e\, \phi \left(R^n{}_a - \frac{1}{2} R\, e_a{}^n\right) \\ & - 2e \left(\nabla_a \nabla^n \phi - e_a{}^n \nabla^k \nabla_k \phi\right)\end{aligned} \tag{B.2}$$

$$\{e_b{}^k, \Pi^n{}_a(e)\} = -\, e_a{}^k\, e_b{}^n \tag{B.3}$$

$$\{e, \Pi^n{}_a(e)\} = +\, e\, e_a{}^n \tag{B.4}$$

$$\left\{\frac{1}{e}, \Pi^n{}_a(e)\right\} = -\, \frac{e_a{}^n}{e} \tag{B.5}$$

$$\{g_{kl}, \Pi^n{}_a(e)\} = +\, 2\ \delta^n_{(k}\, e_{l)a} \tag{B.6}$$

$$\left\{g^{kl}, \Pi^n{}_a(e)\right\} = -\, 2\ g^{n(k}\, e_a{}^{l)} \tag{B.7}$$

$$\{H_{\text{Ham}}[\phi], e\} = +\, \frac{\phi \Pi(e)}{6} \tag{B.8}$$

$$\left\{H_{\text{Ham}}[\phi], \frac{1}{e}\right\} = -\, \frac{\phi \Pi(e)}{6e^2} \tag{B.9}$$

$$\{H_{\text{Ham}}[\phi], e_b{}^k\} = +\, \frac{\phi}{2e} \Pi_{ab}(e)\ e_a{}^k - \frac{\phi}{6e} \Pi(e)\, e_b{}^k \tag{B.10}$$

$$\{H_{\text{Ham}}[\phi], g^{rm}\} = +\, \frac{\phi}{e}\, e^{a(r}\, \Pi^{m)}{}_a(e)\ \ - \frac{\phi}{3e} \Pi(e)\, g^{rm} \tag{B.11}$$

$$\{H_{\text{Ham}}[\phi], M^{MN}\} = +\, \frac{6}{e} \phi\, \Pi^{MN}(M) \tag{B.12}$$

$$\tag{B.13}$$

$$\{L[\gamma], e_a{}^k\} = +\gamma_{ab}\, e^{bk} \tag{B.14}$$

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